

# Sum Divisor Cordial Labeling in the Context of Duplication of Graph Elements

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## Abstract

A sum divisor cordial labeling of a graph  $G$  with vertex set  $V(G)$  is a bijection  $f : V(G) \rightarrow \{1, 2, 3, \dots, |V(G)|\}$  such that an edge  $e = uv$  is assigned the label 1 if  $2 \mid [f(u) + f(v)]$  and 0 otherwise, then the number of edges labeled with 0 and the number of edges labeled with 1 differ by at most 1. If a graph admits a sum divisor cordial labeling, then it is called sum divisor cordial graph. In this paper we have derived some result on sum divisor cordial labeling for the graphs resulted from the duplication of graph elements.

**Keywords:** Sum divisor cordial labeling, Duplication of graph elements.

**AMS Subject Classification(2010):** 05C78.

## 1 Introduction

Throughout this work, by a graph we mean a simple, finite, undirected graph  $G = (V, E)$  of order  $p$  and size  $q$ . For terms and notations related to graph theory which are not defined here, we refer to Gross and Yellen[5] and for standard terminology and notations related to number theory we refer to Burton[2]. This paper includes the results on sum divisor cordial labeling, which is a particular type of graph labeling. The concept of graph labeling was introduced by Rosa.

## 1.1 Definitions

**Definition 1.1** ([8]). *If the vertices or edges or both of the graph are assigned valued subject to certain conditions it is known as graph labeling.*

For a dynamic survey on various graph labeling problems along with an extensive bibliography we refer to Gallian[4].

Cordial labeling was introduced by Cahit as a weaker version of graceful and harmonious labeling of graphs. Combining the concept of divisibility from number theory and cordial labeling from graph labeling, Varatharajan et al.[9] introduced the concept of divisor cordial labeling of a graph.

**Definition 1.2** ([9]). *A bijection  $f : V(G) \rightarrow \{1, 2, \dots, p\}$  is said to be divisor cordial labeling of a graph  $G$  if the induced function  $f^* : E(G) \rightarrow \{0, 1\}$  defined by*

$$f^*(e = uv) = \begin{cases} 1; & \text{if } f(u) \mid f(v) \text{ or } f(v) \mid f(u) \\ 0; & \text{otherwise} \end{cases}$$

*satisfies the condition  $|e_f(0) - e_f(1)| \leq 1$ .*

*A graph with a divisor cordial labeling is called a divisor cordial graph.*

Varatharajan et al.[9] proved that the graphs such as path, cycle, wheel, star, some complete bipartite graphs, some special classes of graphs such as full binary tree, dragon, corona,  $G * K_{2,n}$  and  $G * K_{3,n}$  are divisor cordial. Ghodasara and Adalja[3] derived divisor cordial labeling for ringsum of some standard graphs with star graph.

**Definition 1.3** ([6]). *Let  $f : V(G) \rightarrow \{1, 2, 3, \dots, |V(G)|\}$  be a bijection and let the induced function  $f^* : E(G) \rightarrow \{0, 1\}$  be defined as*

$$f^*(e = uv) = \begin{cases} 1; & \text{if } 2 \mid [f(u) + f(v)] \\ 0; & \text{otherwise.} \end{cases}$$

*Then  $f$  is called sum divisor cordial labeling of graph  $G$  if  $|e_f(0) - e_f(1)| \leq 1$ .*

*A graph with a sum divisor cordial labeling is called sum divisor cordial graph.*

Lourdusamy et al.[6] introduced the concept of sum divisor cordial labeling of graphs. In [7] the same authors have proved that shadow graph and splitting graph of  $K_{1,n}$ , shadow graph, subdivision graph, splitting graph and degree splitting graph of  $B_{n,n}$ , corona of ladder and triangular ladder with  $K_1$ , closed helm are sum divisor cordial graphs. In[1] Adalja and Ghodasara derived sum divisor cordial labeling of cycle, cycle with one chord, cycle with twin chords, cycle with triangle, wheel, helm, web, shell, flower and double fan.

## 2 Sum divisor cordial labeling for duplication of star $K_{1,n}$ related graphs

**Definition 2.1** ([4]). *Duplication of a vertex  $v_k$  of a graph  $G$  produces a new graph  $G'$  by adding a vertex  $v'_k$  with  $N(v_k) = N(v'_k)$ .*

**Theorem 2.1.** *The graph obtained by duplication of any vertex in  $K_{1,n}$  is a sum divisor cordial graph.*

*Proof.* Let  $v_0$  be the apex vertex and  $v_1, v_2, \dots, v_n$  are pendant vertices of  $K_{1,n}$ .

Let  $G$  denote the graph obtained by duplication of any vertex  $v_j$  by a vertex  $v'_j$  in  $K_{1,n}$ .

Depending upon the  $\deg(v_j)$  in  $K_{1,n}$  we have the following two cases.

**Case 1:** Duplication of apex vertex.

The graph obtain by duplication of apex vertex  $v_0$  in  $K_{1,n}$ , which is the complete bipartite graph  $K_{2,n}$ . Hence it is sum divisor cordial graph as proved in [9]

**Case 2:** Duplication of pendant vertex.

The graph obtained by duplication of any pendant vertex in  $K_{1,n}$ , which is again a star graph  $K_{1,n+1}$ , Hence it is sum divisor cordial graph as proved in [9]  $\square$

**Definition 2.2** ([4]). *Duplication of an edge  $e = uv$  of graph  $G$  produces a new graph  $G'$  by adding an edge  $e' = u'v'$  such that  $N(u') = N(u) \cup \{v'\} \setminus \{v\}$  and  $N(v') = N(v) \cup \{u'\} \setminus \{u\}$ .*

**Theorem 2.2.** *The graph obtained by duplication of an edge in  $K_{1,n}$  is sum divisor cordial.*

*Proof.* Let  $v_0$  be the apex vertex and  $v_1, v_2, \dots, v_n$  be consecutive pendant vertices of  $K_{1,n}$ .

Let  $G$  denote the graph obtained by duplication of the edge  $e = v_0v_n$  by a new edge  $e' = v'_0v'_n$  in  $K_{1,n}$ . Hence in  $G$ ,  $\deg(v_0) = n, \deg(v'_0) = n, \deg(v_n) = 1, \deg(v'_n) = 1$  and  $\deg(v_i) = 2, \forall i \in \{1, 2, \dots, n-1\}$ .

Now the resultant graph  $G$  will have  $n+3$  vertices and  $2n$  edges.

We define vertex labeling  $f : V(G) \rightarrow \{1, 2, \dots, n+3\}$  as follows.

$$f(v_0) = 1.$$

$$f(v_1) = 4.$$

$$f(v_n) = 3.$$

$$f(v'_0) = 2.$$

$$f(v'_n) = 5.$$

$$f(v_i) = 4 + i; \quad 2 \leq i \leq n-1.$$

In view of the above defined labeling pattern, we have  $e_f(1) = n = e_f(0)$ .

Thus  $|e_f(0) - e_f(1)| \leq 1$ .

Hence, the graph obtained by duplication of an edge in  $K_{1,n}$  is sum divisor cordial.  $\square$

**Example 2.1.** The star graph  $K_{1,8}$  and sum divisor cordial labeling of the graph obtained by duplication of an edge in  $K_{1,8}$  are shown in Figure 1.

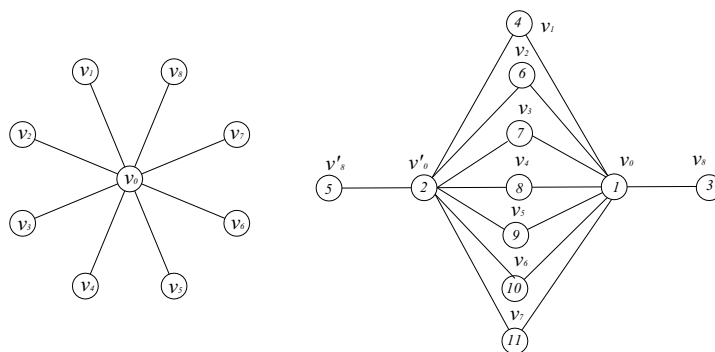


Figure 1

**Definition 2.3** ([4]). Duplication of a vertex  $v_k$  by a new edge  $e' = v'u'$  in a graph  $G$  produces a new graph  $G'$  such that  $N(v') = \{v_k, u'\}$  and  $N(u') = \{v_k, v'\}$ .

**Theorem 2.3.** The graph obtained by duplication of a vertex by an edge in  $K_{1,n}$  is a sum divisor cordial graph.

*Proof.* Let  $v_0$  be the apex vertex of star graph  $K_{1,n}$  and  $v_1, v_2, \dots, v_n$  are pendant vertices of  $K_{1,n}$ . Let  $G$  denote the graph obtained by duplication of a vertex  $v_j$  by an edge  $v'_j v''_j$  in  $K_{1,n}$ .

We consider the following cases.

**Case 1:** Duplication of apex vertex  $v_0$  by an edge  $v'_0 v''_0$ .

Now the resultant graph  $G$  will have  $n + 3$  vertices and  $n + 3$  edges.

We define  $f : V(G) \rightarrow \{1, 2, 3, \dots, n + 2, n + 3\}$  as follows.

$$f(v_0) = 1.$$

$$f(v_1) = 3.$$

$$f(v'_0) = 4.$$

$$f(v''_0) = 2.$$

$$f(v_i) = 3 + i; \quad 2 \leq i \leq n.$$

The following table describes the results of edge labels obtained due to the above labeling pattern.

Cases of $n$	Results for edge labels
$n$ is even	$e_f(0) = \frac{n+2}{2}, e_f(1) = \frac{n+4}{2}$
$n$ is odd	$e_f(1) = \frac{n+3}{2} = e_f(0)$

Thus  $|e_f(0) - e_f(1)| \leq 1$ .

**Case 2:** Duplication of pendant vertex  $v_j$  by an edge  $v'_j v''_j$ .

Without loss of generality we assume that  $v_j = v_n$ . Then in  $G$  we have a cycle of length three having vertices  $v_n, v'_n$  and  $v''_n$ .

Now the resultant graph  $G$  will have  $n + 3$  vertices and  $n + 3$  edges.

We define  $f : V(G) \rightarrow \{1, 2, 3, \dots, n + 2, n + 3\}$  as follows.

$$\begin{aligned}
 f(v_0) &= 1. \\
 f(v_n) &= 3. \\
 f(v'_n) &= 4. \\
 f(v''_n) &= 2. \\
 f(v_i) &= 4 + i; \quad 1 \leq i \leq n - 1.
 \end{aligned}$$

The following table describes the results of edge labels obtained due to the above labeling pattern.

Cases of $n$	Results for edge labels
$n$ is even	$e_f(0) = \frac{n+2}{2}, e_f(1) = \frac{n+4}{2}$
$n$ is odd	$e_f(1) = \frac{n+3}{2} = e_f(0)$

Thus  $|e_f(0) - e_f(1)| \leq 1$ .

Hence, the graph obtained by duplication of vertex by edge in  $K_{1,n}$  is sum divisor cordial.  $\square$

**Example 2.2.** The star graph  $K_{1,5}$  and sum divisor cordial labeling of the graph obtained by duplication of apex vertex  $v_0$  by an edge  $e = v'_0 v''_0$  in  $K_{1,5}$  are shown in Figure 2.

**Example 2.3.** The star graph  $K_{1,7}$  and sum divisor cordial labeling of the graph obtained by duplication of vertex  $v_7$  by an edge  $e = v'_7 v''_7$  in  $K_{1,7}$  are shown in Figure 3.

**Definition 2.4** ([4]). Duplication of an edge  $e = uv$  by a new vertex  $v'$  in a graph  $G$  produces a new graph  $G'$  such that  $N(v') = \{u, v\}$ .

**Theorem 2.4.** The graph obtained by duplication of an edge by a vertex in  $K_{1,n}$  is a sum divisor cordial graph.

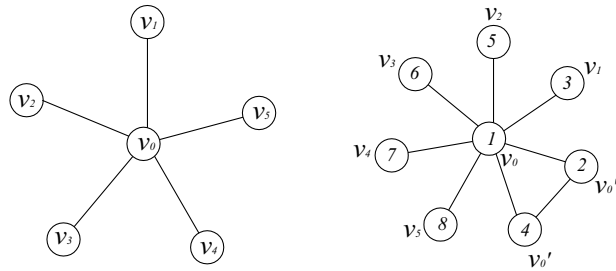


Figure 2

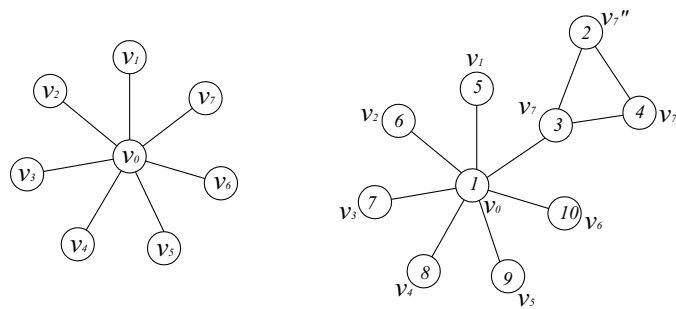


Figure 3

*Proof.* Let  $v_0$  be the apex vertex and  $v_1, v_2, \dots, v_n$  be the consecutive pendant vertices of  $K_{1,n}$ . Let  $G$  denote the graph obtained by duplication of the edge  $v_0v_1$  by a vertex  $v_1'$ . Now the resultant graph  $G$  will have  $n + 2$  vertices and  $n + 2$  edges. We define  $f : V(G) \rightarrow \{1, 2, \dots, n + 2\}$  as follows.

$$f(v_0) = 1.$$

$$f(v_1) = 2.$$

$$f(v_1') = 4.$$

$$f(v_i) = 2i - 1, \quad 2 \leq i \leq 3$$

$$f(v_i) = 2 + i; \quad 4 \leq i \leq n.$$

The following table describes the results of edge labels obtained due to the above labeling pattern.

Cases of $n$	Results for edge labels
$n$ is odd	$e_f(0) = \frac{n+1}{2}, e_f(1) = \frac{n+3}{2}$
$n$ is even	$e_f(1) = \frac{n+2}{2} = e_f(0)$

Thus  $|e_f(0) - e_f(1)| \leq 1$ .

Hence, the graph obtained by duplication of edge by vertex in  $K_{1,n}$  is sum divisor cordial.  $\square$

**Example 2.4.** The star graph  $K_{1,6}$  and sum divisor cordial labeling of the graph obtained by duplication of edge  $e = v_0v_1$  by vertex  $v'_1$  in  $K_{1,6}$  are shown in Figure 4.

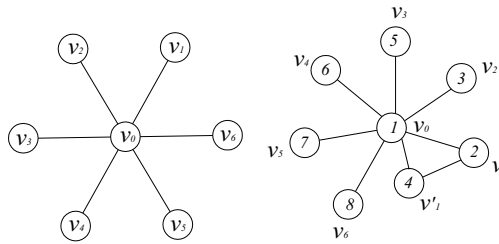


Figure 4

### 3 Sum divisor cordial labeling for duplication of cycle $C_n$ related graphs

**Theorem 3.1.** The graph obtained by duplication of an arbitrary vertex  $v_k$  in cycle  $C_n$  is a sum divisor cordial graph.

*Proof.* Let  $v_1, v_2, \dots, v_n$  be the vertices of cycle  $C_n$ . Without loss of generality we duplicate the vertex  $v_1$  by the vertex  $v'_1$ . Now the resultant graph  $G$  will have  $n + 1$  vertices and  $n + 2$  edges. We define  $f : V(G) \rightarrow \{1, 2, 3, \dots, n + 1\}$  as follows.

For  $n \equiv 0, 1, 3(mod 4)$  :

$$f(v'_1) = n + 1.$$

$$f(v_i) = \begin{cases} i & ; i \equiv 1, 0(mod 4) \\ i + 1 & ; i \equiv 2(mod 4) \\ i - 1 & ; i \equiv 3(mod 4); \quad 1 \leq i \leq n. \end{cases}$$

For  $n \equiv 2(mod 4)$  :

$$f(v'_1) = n.$$

$$f(v_i) = \begin{cases} i & ; i \equiv 1, 0(mod 4) \\ i + 1 & ; i \equiv 2(mod 4) \\ i - 1 & ; i \equiv 3(mod 4); \quad 1 \leq i \leq n. \end{cases}$$

Now  $2 \mid [f(u) + f(v)]$  where  $f(u)$  and  $f(v)$  both are of same parity.

From the above defined labeling, the vertex labels for two consecutive vertices in the graph are arranged in a pattern such that

1.  $f(v_{4t-3}), f(v_{4t-2})$  are odd and  $f(v_{4t-1}), f(v_{4t})$  are even or
2.  $f(v_{4t-3}), f(v_{4t-2})$  are even and  $f(v_{4t-1}), f(v_{4t})$  are odd and so on.

Hence  $f(v_{4t-3}v_{4t-2}) = 1, f(v_{4t-2}v_{4t-1}) = 0, f(v_{4t-1}v_{4t}) = 1, f(v_{4t}v_{4t+1}) = 0$ , where  $1 \leq t \leq \frac{|E|}{4}$ .

The following table describes the results of edge labels obtained due to the above labeling pattern.

Cases of $n$	Results for edge labels
$n$ is odd	$e_f(1) = \frac{n+1}{2}, e_f(0) = \frac{n+3}{2}$
$n$ is even	$e_f(1) = \frac{n+2}{2} = e_f(0)$

Thus  $|e_f(0) - e_f(1)| \leq 1$ .

Hence, the graph obtained by duplication of an arbitrary vertex in  $C_n$  is sum divisor cordial.  $\square$

**Example 3.1.** The cycle graph  $C_5$  and sum divisor cordial labeling of the graph obtained by duplication an arbitrary vertex  $v_1$  by vertex  $v'_1$  in  $C_5$  are shown in Figure 5.

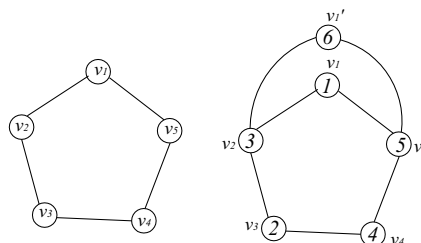


Figure 5



**Theorem 3.2.** *The graph obtained by duplication of an arbitrary edge  $e_k$  in cycle  $C_n$  is a sum divisor cordial graph.*

*Proof.* Let  $v_1, v_2, \dots, v_n$  be the vertices of cycle  $C_n$  and  $e_1, e_2, \dots, e_n$  be the edges of cycle  $C_n$ . Without loss of generality we duplicate the edge  $e_1 = v_1v_2$  thus added vertices are  $v'_1$  and  $v'_2$  such that  $N(v'_1) = \{v'_2, v_n\}$  and  $N(v'_2) = \{v'_1, v_3\}$ . Now the resultant graph  $G$  will have  $n + 2$  vertices and  $n + 3$  edges.

We define  $f : V(G) \rightarrow \{1, 2, 3, \dots, n + 2\}$  as follows.

For  $n \equiv 0(\text{mod}4)$  :

$$\begin{aligned} f(v'_1) &= n + 1. \\ f(v'_2) &= n + 2. \\ f(v_i) &= \begin{cases} i & ; i \equiv 1, 0(\text{mod} 4) \\ i + 1 & ; i \equiv 2(\text{mod} 4) \\ i - 1 & ; i \equiv 3(\text{mod} 4); \quad 1 \leq i \leq n. \end{cases} \end{aligned}$$

For  $n \equiv 2(\text{mod}4)$  :

$$\begin{aligned} f(v'_1) &= n. \\ f(v'_2) &= n + 2. \\ f(v_i) &= \begin{cases} i & ; i \equiv 1, 0(\text{mod} 4) \\ i + 1 & ; i \equiv 2(\text{mod} 4) \\ i - 1 & ; i \equiv 3(\text{mod} 4); \quad 1 \leq i \leq n - 1. \end{cases} \\ f(v_n) &= n + 1. \end{aligned}$$

For  $n \equiv 1(\text{mod}4)$  :

$$\begin{aligned} f(v'_1) &= 1. \\ f(v'_2) &= 3. \\ f(v_i) &= \begin{cases} i + 1 & ; i \equiv 1(\text{mod} 4) \\ i + 2 & ; i \equiv 2, 3(\text{mod} 4) \\ i + 3 & ; i \equiv 0(\text{mod} 4); \quad 1 \leq i \leq n - 2. \end{cases} \\ f(v_{n-1}) &= n + 1. \\ f(v_n) &= n + 2. \end{aligned}$$

For  $n \equiv 3(\text{mod}4)$  :

$$\begin{aligned}
 f(v'_1) &= 1. \\
 f(v'_2) &= 3. \\
 f(v_i) &= \begin{cases} i+1 & ; i \equiv 1(\text{mod}4) \\ i+2 & ; i \equiv 2,3(\text{mod}4) \\ i+3 & ; i \equiv 0(\text{mod}4); \quad 1 \leq i \leq n-2. \end{cases} \\
 f(v_{n-1}) &= n+2. \\
 f(v_n) &= n+1.
 \end{aligned}$$

The following table describes the results of edge labels obtained due to the above labeling pattern.

Cases of $n$	Results for edge labels
$n$ is even	$e_f(0) = \frac{n+2}{2}, e_f(1) = \frac{n+4}{2}$
$n$ is odd	$e_f(1) = \frac{n+3}{2} = e_f(0)$

Thus  $|e_f(0) - e_f(1)| \leq 1$ .

Hence, the graph obtained by duplication of an arbitrary edge in  $C_n$  is sum divisor cordial.  $\square$

**Example 3.2.** The cycle graph  $C_6$  and sum divisor cordial labeling of the graph obtained by duplication an arbitrary edge in  $C_6$  are shown in Figure 6.

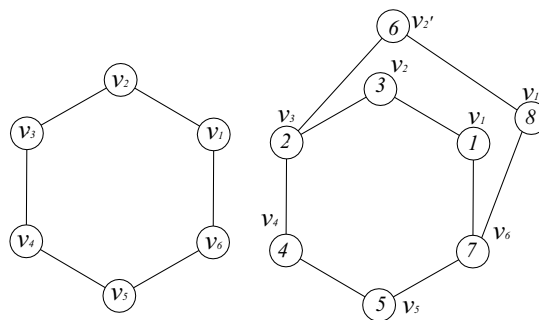


Figure 6

**Theorem 3.3.** The graph obtained by duplication of an arbitrary vertex by a new edge in cycle  $C_n$  is a sum divisor cordial graph, for  $n \equiv 0, 1, 2(\text{mod}4)$ .

*Proof.* Let  $v_1, v_2, \dots, v_n$  be the vertices and  $e_1, e_2, \dots, e_n$  be the edges of cycle  $C_n$ . Without loss of generality we duplicate the vertex  $v_1$  by an edge  $e'$  with end vertices as  $v'_1$  and  $v'_2$ . The

resultant graph  $G$  will have  $n + 2$  vertices and  $n + 3$  edges.

We define  $f : V(G) \rightarrow \{1, 2, 3, \dots, n + 2\}$  as follows.

$$f(v_i) = \begin{cases} i & ; i \equiv 1, 0(\text{mod } 4) \\ i + 1 & ; i \equiv 2(\text{mod } 4) \\ i - 1 & ; i \equiv 3(\text{mod } 4); \quad 1 \leq i \leq n. \end{cases}$$

For  $n \equiv 0, 1(\text{mod } 4)$  :

$$\begin{aligned} f(v'_1) &= n + 1. \\ f(v'_2) &= n + 2. \end{aligned}$$

For  $n \equiv 2(\text{mod } 4)$  :

$$\begin{aligned} f(v'_1) &= n. \\ f(v'_2) &= n + 2. \end{aligned}$$

The following table describes the results of edge labels obtained due to the above labeling pattern.

Cases of $n$	Results for edge labels
$n \equiv 2(\text{mod } 4)$	$e_f(0) = \frac{n+2}{2}, e_f(1) = \frac{n+4}{2}$
$n \equiv 0(\text{mod } 4)$	$e_f(1) = \frac{n+2}{2}, e_f(0) = \frac{n+4}{2}$
$n \equiv 1(\text{mod } 4)$	$e_f(1) = \frac{n+3}{2} = e_f(0)$

Thus  $|e_f(0) - e_f(1)| \leq 1$ .

Hence, the graph obtained by duplication of an arbitrary vertex by a new edge in  $C_n$  is sum divisor cordial. □

**Example 3.3.** *The cycle graph  $C_5$  and sum divisor cordial labeling of the graph obtained by duplication an arbitrary vertex  $v_1$  by a new edge  $v'_1v'_2$  in  $C_5$  are shown in Figure 7.*

**Theorem 3.4.** *The graph obtained by duplication of all the vertices by new edge in cycle  $C_n$  is a sum divisor cordial graph.*

*Proof.* Let  $v_1, v_2, \dots, v_n$  be vertices and  $e_1, e_2, \dots, e_n$  be edges of cycle  $C_n$ . Let the graph obtained by duplicating all the vertices by edges in cycle  $C_n$  is  $G$ . The resultant graph  $G$  will have  $3n$  vertices and  $4n$  edges.

We define  $f : V(G) \rightarrow \{1, 2, 3, \dots, 3n\}$  as follows.

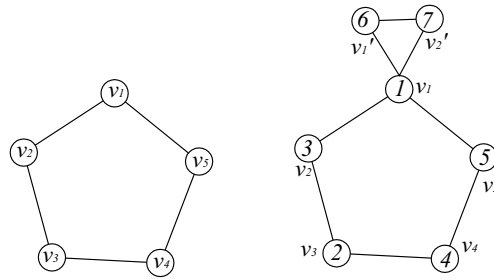


Figure 7

For  $n \equiv 0, 1, 3(\text{mod } 4)$  :

$$f(v_i) = \begin{cases} i & ; i \equiv 1, 0(\text{mod } 4) \\ i + 1 & ; i \equiv 2(\text{mod } 4) \\ i - 1 & ; i \equiv 3(\text{mod } 4); \quad 1 \leq i \leq n. \end{cases}$$

For  $n \equiv 0(\text{mod } 4)$  :

$$f(v'_i) = \begin{cases} 2i - 1 + n & ; i \equiv 1, 3(\text{mod } 4) \\ 2i - 2 + n & ; i \equiv 0, 2(\text{mod } 4); \quad 1 \leq i \leq \frac{n}{2}. \end{cases}$$

$$f(v''_i) = f(v'_i) + 2; \quad 1 \leq i \leq \frac{n}{2}.$$

$$f(v'_i) = n + 2i - 1; \quad \frac{n}{2} + 1 \leq i \leq n.$$

$$f(v''_i) = n + 2i; \quad \frac{n}{2} + 1 \leq i \leq n.$$

For  $n \equiv 3(\text{mod } 4)$  :

$$f(v'_i) = \begin{cases} 2i - 1 + n & ; i \equiv 1, 3(\text{mod } 4) \\ 2i - 2 + n & ; i \equiv 0, 2(\text{mod } 4); \quad 1 \leq i \leq \frac{n+1}{2}. \end{cases}$$

$$f(v''_i) = f(v'_i) + 2; \quad 1 \leq i \leq \frac{n+1}{2}.$$

$$f(v'_i) = n + 2i - 1; \quad \frac{n+1}{2} + 1 \leq i \leq n.$$

$$f(v''_i) = n + 2i; \quad \frac{n+1}{2} + 1 \leq i \leq n.$$

For  $n \equiv 1(mod 4)$  :

$$\begin{aligned}
 f(v'_i) &= \begin{cases} 2i + n & ; i \equiv 1, 3(mod 4) \\ 2i - 3 + n & ; i \equiv 0, 2(mod 4); \quad 1 \leq i \leq \frac{n-1}{2}. \end{cases} \\
 f(v''_i) &= f(v'_i) + 2; \quad 1 \leq i \leq \frac{n-1}{2}. \\
 f(v'_i) &= n + 2i - 1; \quad \frac{n-1}{2} + 1 \leq i \leq n. \\
 f(v''_i) &= n + 2i; \quad \frac{n-1}{2} + 1 \leq i \leq n.
 \end{aligned}$$

For  $n \equiv 2(mod 4)$  :

$$\begin{aligned}
 f(v_i) &= \begin{cases} i + 1 & ; i \equiv 1(mod 4) \\ i + 2 & ; i \equiv 2(mod 4) \\ i - 2 & ; i \equiv 3(mod 4) \\ i - 1 & ; i \equiv 0(mod 4); \quad 1 \leq i \leq n - 1. \end{cases} \\
 f(v_n) &= n + 2. \\
 f(v'_1) &= n - 1. \\
 f(v''_1) &= n + 1. \\
 f(v'_i) &= \begin{cases} 2i - 3 + n & ; i \equiv 1, 3(mod 4) \\ 2i + n & ; i \equiv 0, 2(mod 4); \quad 2 \leq i \leq \frac{n}{2}. \end{cases} \\
 f(v''_i) &= f(v'_i) + 2; \quad 2 \leq i \leq \frac{n}{2}. \\
 f(v'_i) &= n + 2i - 1; \quad \frac{n}{2} + 1 \leq i \leq n. \\
 f(v''_i) &= n + 2i; \quad \frac{n}{2} + 1 \leq i \leq n.
 \end{aligned}$$

In view of the above defined labeling pattern, we have  $e_f(1) = 2n = e_f(0)$ .

Thus  $|e_f(0) - e_f(1)| \leq 1$ .

Hence, the graph obtained by duplication of all the vertices by new edge in cycle  $C_n$  is sum divisor cordial. □

**Example 3.4.** *The cycle graph  $C_5$  and sum divisor cordial labeling of the graph obtained by duplication of all the vertices by new edge in cycle  $C_5$  are shown in Figure 8.*

**Theorem 3.5.** *The graph obtained by duplication of an arbitrary edge by a new vertex in cycle  $C_n$  is a sum divisor cordial graph.*

*Proof.* Let  $v_1, v_2, \dots, v_n$  be the vertices and  $e_1, e_2, \dots, e_n$  be the edges of cycle  $C_n$ . Without loss of generality we duplicate the edge  $v_1v_n$  by vertex  $v'$ . The resultant graph  $G$  will have  $n + 1$

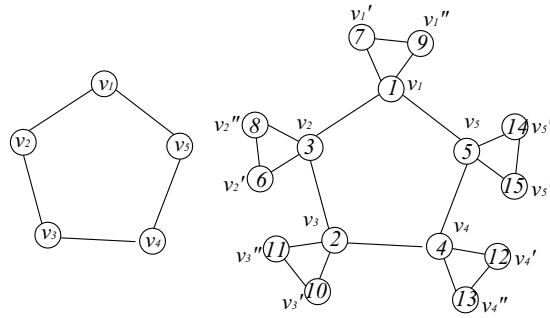


Figure 8

vertices and  $n + 2$  edges.

We define  $f : V(G) \rightarrow \{1, 2, 3, \dots, n + 1\}$  as follows.

For  $n \equiv 0, 1, 3(mod 4)$  :

$$f(v_i) = \begin{cases} i & ; i \equiv 1, 0(mod 4) \\ i + 1 & ; i \equiv 2(mod 4) \\ i - 1 & ; i \equiv 3(mod 4); \quad 1 \leq i \leq n. \end{cases}$$

$$f(v') = n + 1.$$

For  $n \equiv 2(mod 4)$  :

$$f(v_i) = \begin{cases} i & ; i \equiv 1, 0(mod 4) \\ i + 1 & ; i \equiv 2(mod 4) \\ i - 1 & ; i \equiv 3(mod 4); \quad 1 \leq i \leq n - 1. \end{cases}$$

$$f(v_n) = n + 1.$$

$$f(v') = n.$$

The following table describes the results of edge labels obtained due to the above labeling pattern.

Cases of $n$	Results for edge labels
$n$ is odd	$e_f(1) = \frac{n+1}{2}, e_f(0) = \frac{n+3}{2}$
$n$ is even	$e_f(1) = \frac{n+2}{2} = e_f(0)$

Thus  $|e_f(0) - e_f(1)| \leq 1$ .

Hence, the graph obtained by duplication of an arbitrary edge by a new vertex in  $C_n$  is sum divisor cordial. □

**Example 3.5.** The cycle graph  $C_7$  and sum divisor cordial labeling of the graph obtained by duplication an arbitrary edge by a new vertex in  $C_7$  are shown in Figure 9.

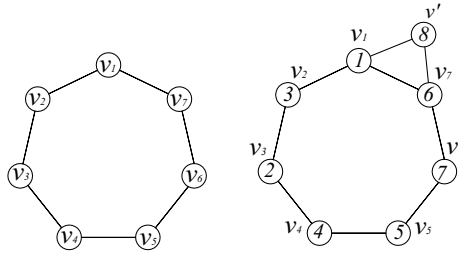


Figure 9

**Remark 3.1.** In the section 3, Cycle related Theorems, the proof for the values of  $e_f(1)$  and  $e_f(0)$  is same as given in Theorem 3.1.

### 4 Sum divisor cordial labeling for duplication of path $P_n$ related graphs

**Theorem 4.1.** The graph obtained by duplication of an arbitrary vertex  $v_k$  in path  $P_n$  is a sum divisor cordial graph.

*Proof.* Let  $v_1, v_2, \dots, v_n$  be the vertices of path  $P_n$ . Let  $v_k$  be the vertex duplicated by new vertex  $v'_k$ ,  $1 \leq k \leq n$ . Then the resultant graph  $G$  will have  $n + 1$  vertices and if  $k = 1$  or  $k = n$  then  $n$  edges, if  $k \neq 1$  or  $k \neq n$  then  $n + 1$  edges.

We define  $f : V(G) \rightarrow \{1, 2, 3, \dots, n + 1\}$  as follows.

$$f(v_i) = \begin{cases} i & ; i \equiv 1, 0(\text{mod } 4) \\ i + 1 & ; i \equiv 2(\text{mod } 4) \\ i - 1 & ; i \equiv 3(\text{mod } 4); \quad 1 \leq i \leq n. \end{cases}$$

For  $n \equiv 0, 1, 3(\text{mod } 4)$

$$f(v'_k) = n + 1.$$

For  $n \equiv 2(\text{mod } 4)$

$$f(v'_k) = n.$$

When either of pendant vertex is duplicated, the following table describes the results of edge labels obtained due to the above labeling pattern.

Cases of $n$	Results for edge labels
$n$ is odd	$e_f(1) = \frac{n-1}{2}, e_f(0) = \frac{n+1}{2}$
$n$ is even	$e_f(0) = \frac{n}{2}, e_f(1) = \frac{n}{2}$

When a vertex other than pendant vertex is duplicated, the following table describes the results of edge labels obtained due to the above labeling pattern.

Cases of $n$	Results for edge labels
$n$ is odd	$e_f(1) = \frac{n+1}{2}, e_f(0) = \frac{n+1}{2}$
$n$ is even	$e_f(0) = \frac{n}{2}, e_f(1) = \frac{n+2}{2}$

Thus  $|e_f(0) - e_f(1)| \leq 1$ .

Hence, the graph obtained by duplication of an arbitrary vertex in  $P_n$  is sum divisor cordial. □

**Example 4.1.** The path graph  $P_5$  and sum divisor cordial labeling of the graph obtained by duplication an arbitrary vertex in  $P_5$  are shown in Figure 10.

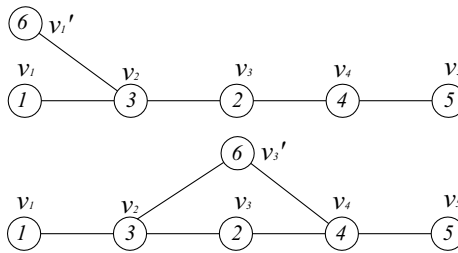


Figure 10

**Theorem 4.2.** The graph obtained by duplication of an arbitrary edge  $e_k$  in  $P_n$  is a sum divisor cordial graph.

*Proof.* Let  $e_1, e_2, \dots, e_{n-1}$  be the edges of path  $P_n$ .

Let the duplicated edge be  $v_k v_{k+1}$ ,  $1 \leq k \leq n - 1$  and let  $v'_1$  and  $v'_2$  be newly added vertices such that

$$N(v'_1) = \begin{cases} \{v'_2\} & ; k = 1 \\ \{v'_2 v_{k-1}\} & ; k \neq 1. \end{cases}$$

$$N(v'_2) = \{v'_1, v_{k+2}\}.$$

Then the resultant graph  $G$  will have  $n + 2$  vertices and if  $k = 1$  then  $n + 1$  edges, if  $k \neq 1$  then  $n + 2$  edges.



We define  $f : V(G) \rightarrow \{1, 2, 3, \dots, n + 2\}$  as follows.

$$f(v_i) = \begin{cases} i & ; i \equiv 1, 0(\text{mod } 4) \\ i + 1 & ; i \equiv 2(\text{mod } 4) \\ i - 1 & ; i \equiv 3(\text{mod } 4); \quad 1 \leq i \leq n. \end{cases}$$

For  $n \equiv 0, 1, 3(\text{mod } 4)$

$$f(v'_1) = n + 2.$$

$$f(v'_2) = n + 1.$$

For  $n \equiv 2(\text{mod } 4)$

$$f(v'_1) = n + 1.$$

$$f(v'_2) = n + 2.$$

When either of the pendant edge is duplicated, the following table describes the results of edge labels obtained due to the above labeling pattern.

Cases of $n$	Results for edge labels
$n$ is odd	$e_f(1) = \frac{n+1}{2} = e_f(0)$
$n$ is even	$e_f(1) = \frac{n}{2}, e_f(0) = \frac{n+2}{2}$

When an edge other than a pendant edge is duplicated, the following table describes the results of edge labels obtained due to the above labeling pattern.

Cases of $n$	Results for edge labels
$n$ is odd	$e_f(1) = \frac{n+3}{2}, e_f(0) = \frac{n+1}{2}$
$n$ is even	$e_f(1) = \frac{n+2}{2} = e_f(0)$

Thus  $|e_f(0) - e_f(1)| \leq 1$ .

Hence, the graph obtained by duplication of an arbitrary edge in  $P_n$  is sum divisor cordial.  $\square$

**Example 4.2.** The path graph  $P_5$  and sum divisor cordial labeling of the graph obtained by duplication an arbitrary edge in  $P_5$  are shown in Figure 11.

**Theorem 4.3.** The graph obtained by duplication of an arbitrary vertex by a new edge in path  $P_n$  is a sum divisor cordial graph.

*Proof.* Let  $v_1, v_2, \dots, v_n$  be the vertices and  $e_1, e_2, \dots, e_{n-1}$  be the edges of path  $P_n$ . Duplicate the vertex  $v_k$  by an edge  $e'$  with end vertices as  $v'_k$  and  $v''_k$ . The resultant graph  $G$  will have

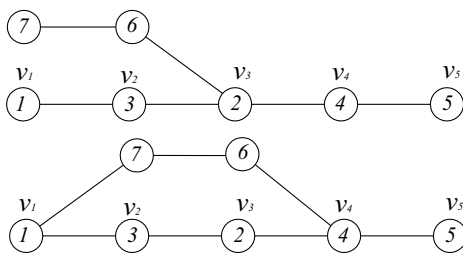


Figure 11

$n + 2$  vertices and  $n + 2$  edges.

We define  $f : V(G) \rightarrow \{1, 2, 3, \dots, n + 2\}$  as follows.

$$f(v_i) = \begin{cases} i & ; i \equiv 1, 0(\text{mod } 4) \\ i + 1 & ; i \equiv 2(\text{mod } 4) \\ i - 1 & ; i \equiv 3(\text{mod } 4); \quad 1 \leq i \leq n. \end{cases}$$

For  $n \equiv 0, 1, 3(\text{mod } 4)$

$$\begin{aligned} f(v'_k) &= n + 1. \\ f(v''_k) &= n + 2. \end{aligned}$$

For  $n \equiv 2(\text{mod } 4)$

$$\begin{aligned} f(v'_k) &= n. \\ f(v''_k) &= n + 2. \end{aligned}$$

The following table describes the results of edge labels obtained due to the above labeling pattern.

Cases of $n$	Results for edge labels
$n$ is even	$e_f(0) = \frac{n+2}{2} = e_f(1)$
$n$ is odd	$e_f(1) = \frac{n+1}{2}, e_f(0) = \frac{n+3}{2}$

Thus  $|e_f(0) - e_f(1)| \leq 1$ .

Hence, the graph obtained by duplication of an arbitrary vertex in  $P_n$  is sum divisor cordial.  $\square$

**Example 4.3.** The path graph  $P_5$  and sum divisor cordial labeling of the graph obtained by duplication an arbitrary vertex by a new edge in  $P_5$  are shown in Figure 12.

**Theorem 4.4.** The graph obtained by duplication of an arbitrary edge by a new vertex in path  $P_n$  is a sum divisor cordial graph.

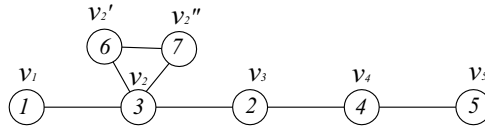


Figure 12

*Proof.* Let  $v_1, v_2, \dots, v_n$  be the vertices and  $e_1, e_2, \dots, e_{n-1}$  be the edges of path  $P_n$ . Duplicate the edge  $e = v_k v_{k+1}$  by a vertex  $v'$ . The resultant graph  $G$  will have  $n + 1$  vertices and  $n + 1$  edges.

We define  $f : V(G) \rightarrow \{1, 2, 3, \dots, n + 1\}$  as follows.

$$f(v_i) = \begin{cases} i & ; i \equiv 1, 0(\text{mod } 4) \\ i + 1 & ; i \equiv 2(\text{mod } 4) \\ i - 1 & ; i \equiv 3(\text{mod } 4); \quad 1 \leq i \leq n. \end{cases}$$

For  $n \equiv 0, 1, 3(\text{mod } 4)$

$$f(v') = n + 1.$$

For  $n \equiv 2(\text{mod } 4)$

$$f(v') = n.$$

The following table describes the results of edge labels obtained due to the above labeling pattern.

Cases of $n$	Results for edge labels
$n$ is odd	$e_f(0) = \frac{n+1}{2} = e_f(1)$
$n$ is even	$e_f(0) = \frac{n}{2}, e_f(1) = \frac{n+2}{2}$

Thus  $|e_f(0) - e_f(1)| \leq 1$ .

Hence, the graph obtained by duplication of an arbitrary edge by a new vertex in  $P_n$  is sum divisor cordial. □

**Example 4.4.** The path graph  $P_5$  and sum divisor cordial labeling of the graph obtained by duplication an arbitrary edge by a new vertex in  $P_5$  are shown in Figure 13.

**Theorem 4.5.** The graph obtained by duplication of all the vertices by edges in path  $P_n$  is a sum divisor cordial graph.

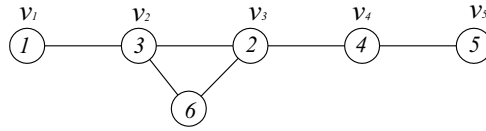


Figure 13

*Proof.* Let  $v_1, v_2, \dots, v_n$  be the vertices and  $e_1, e_2, \dots, e_{n-1}$  be the edges of path  $P_n$ .

Let  $G$  denote the graph obtained by duplicating all the vertices by edges in path  $P_n$ . The resultant graph  $G$  will have  $3n$  vertices and  $4n - 1$  edges. Let the edge so added corresponding to vertex  $v_i$  has end vertices as  $v'_i$  and  $v''_i$ , where  $1 \leq i \leq n$ .

We define  $f : V(G) \rightarrow \{1, 2, 3, \dots, 3n\}$  as follows.

For  $n \equiv 0, 1, 3(\text{mod } 4)$  :

$$f(v_i) = \begin{cases} i & ; i \equiv 1, 0(\text{mod } 4) \\ i + 1 & ; i \equiv 2(\text{mod } 4) \\ i - 1 & ; i \equiv 3(\text{mod } 4); \quad 1 \leq i \leq n. \end{cases}$$

For  $n \equiv 0(\text{mod } 4)$  :

$$f(v'_i) = \begin{cases} 2i - 1 + n & ; i \equiv 1, 3(\text{mod } 4) \\ 2i - 2 + n & ; i \equiv 0, 2(\text{mod } 4); \quad 1 \leq i \leq \frac{n}{2}. \end{cases}$$

$$f(v''_i) = f(v'_i) + 2; \quad 1 \leq i \leq \frac{n}{2}.$$

$$f(v'_i) = n + 2i - 1; \quad \frac{n}{2} + 1 \leq i \leq n.$$

$$f(v''_i) = n + 2i; \quad \frac{n}{2} + 1 \leq i \leq n.$$

For  $n \equiv 3(\text{mod } 4)$  :

$$f(v'_i) = \begin{cases} 2i - 1 + n & ; i \equiv 1, 3(\text{mod } 4) \\ 2i - 2 + n & ; i \equiv 0, 2(\text{mod } 4); \quad 1 \leq i \leq \frac{n+1}{2}. \end{cases}$$

$$f(v''_i) = f(v'_i) + 2, \quad 1 \leq i \leq \frac{n+1}{2}.$$

$$f(v'_i) = n + 2i - 1; \quad \frac{n+1}{2} + 1 \leq i \leq n.$$

$$f(v''_i) = n + 2i; \quad \frac{n+1}{2} + 1 \leq i \leq n.$$

For  $n \equiv 1(mod 4)$  :

$$\begin{aligned}
 f(v'_i) &= \begin{cases} 2i + n & ; i \equiv 1, 3(mod 4) \\ 2i - 3 + n & ; i \equiv 0, 2(mod 4); \quad 1 \leq i \leq \frac{n-1}{2}. \end{cases} \\
 f(v''_i) &= f(v'_i) + 2; \quad 1 \leq i \leq \frac{n-1}{2}. \\
 f(v'_i) &= n + 2i - 1; \quad \frac{n-1}{2} + 1 \leq i \leq n. \\
 f(v''_i) &= n + 2i; \quad \frac{n-1}{2} + 1 \leq i \leq n.
 \end{aligned}$$

For  $n \equiv 2(mod 4)$  :

$$\begin{aligned}
 f(v_i) &= \begin{cases} i + 1 & ; i \equiv 1(mod 4) \\ i + 2 & ; i \equiv 2(mod 4) \\ i - 2 & ; i \equiv 3(mod 4) \\ i - 1 & ; i \equiv 0(mod 4); \quad 1 \leq i \leq n - 1. \end{cases} \\
 f(v_n) &= n + 2. \\
 f(v'_1) &= n - 1. \\
 f(v''_1) &= n + 1. \\
 f(v'_i) &= \begin{cases} 2i - 2 + n & ; i \equiv 1, 3(mod 4) \\ 2i - 1 + n & ; i \equiv 0, 2(mod 4); \quad 2 \leq i \leq \frac{n-1}{2}. \end{cases} \\
 f(v''_i) &= f(v'_i) + 2; \quad 2 \leq i \leq \frac{n-1}{2}. \\
 f(v'_i) &= n + 2i - 1; \quad \frac{n-1}{2} + 1 \leq i \leq n. \\
 f(v''_i) &= n + 2i; \quad \frac{n-1}{2} + 1 \leq i \leq n.
 \end{aligned}$$

The following table describes the results of edge labels obtained due to the above labeling pattern.

Cases of $n$	Results for edge labels
$n \equiv 1, 2, 3(mod 4)$	$e_f(1) = 2n - 1, e_f(0) = 2n$
$n \equiv 0(mod 4)$	$e_f(0) = 2n - 1, e_f(1) = 2n$

Thus  $|e_f(0) - e_f(1)| \leq 1$ .

Hence, the graph obtained by duplication of an arbitrary edge by a new vertex in  $P_n$  is sum divisor cordial. □

**Example 4.5.** *The path graph  $P_5$  and sum divisor cordial labeling of the graph obtained by duplication of all the vertices by edges in  $P_5$  are shown in Figure 14.*

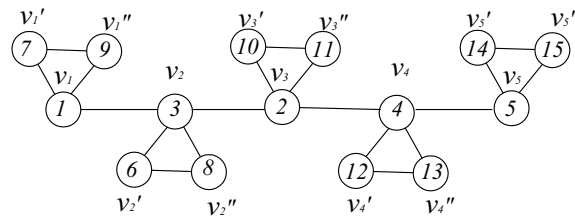


Figure 14

## 5 Concluding Remarks

Here, we have investigated some new results related to the graph operation duplication of graph for sum divisor cordial labeling technique. To explore some new sum divisor cordial graphs in the context of other graph operations is an open area of research.

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