ON \textsc{*-Differential Identities Equipped with Skew Lie Product}

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Abstract. Let $\mathcal{R}$ be a ring with involution $'*$', skew Lie product of $x, y \in \mathcal{R}$ is defined by $\mathsf{V}[x, y] = xy - yx^*$. In this paper, we discuss the commutativity of prime rings with involution equipped with skew Lie product which satisfy the certain \textsc{*-differential identities involving generalized derivations.}

1. Introduction

Throughout the paper, $\mathcal{R}$ always denotes an associative ring with center $\mathcal{Z}(\mathcal{R})$. An additive mapping $*: \mathcal{R} \to \mathcal{R}$ is called an involution if $*$ is an anti-automorphism of order 2; that is, $(x^*)^* = x$ for all $x \in \mathcal{R}$. An element $x$ in a ring with involution is said to be hermitian if $x^* = x$ and skew-hermitian if $x^* = -x$. The sets of all hermitian and skew-hermitian elements of $\mathcal{R}$ will be denoted by $H(\mathcal{R})$ and $S(\mathcal{R})$, respectively. A ring equipped with an involution is known as ring with involution or \textsc{*-ring}. If $\mathcal{R}$ is 2-torsion free, then every $x \in \mathcal{R}$ can be uniquely represented in the form $2x = h + k$, where $h \in H(\mathcal{R})$ and $k \in S(\mathcal{R})$. Note that $S(\mathcal{R}) = H(\mathcal{R})$ if $\text{char} (\mathcal{R}) = 2$. The involution is said to be of the first kind if $\mathcal{Z}(\mathcal{R}) \subseteq H(\mathcal{R})$, otherwise it is said to be of the second kind. In the later case it is worthwhile to see that $S(\mathcal{R}) \cap \mathcal{Z}(\mathcal{R}) \neq \{0\}$. The skew Lie product of $x, y \in \mathcal{R}$ is defined by $\mathsf{V}[x, y] = xy - yx^*$. We refer the reader to [5] and [6] for justification and amplification for the above mentioned notations and key definitions.

An additive mapping $d : \mathcal{R} \to \mathcal{R}$ is said to be a derivation on $\mathcal{R}$ if $d(xy) = d(x)y + xd(y)$ for all $x, y \in \mathcal{R}$. A derivation $d$ is said to be inner if there exists $a \in \mathcal{R}$ such that $d(x) = ax - xa$ for all $x \in \mathcal{R}$. Following Bresar [4], an additive map $\mathfrak{F} : \mathcal{R} \to \mathcal{R}$ is called a generalized derivation of $\mathcal{R}$ if there exists a derivation $d$ of $\mathcal{R}$ such that $\mathfrak{F}(xy) = \mathfrak{F}(x)y + xd(y)$ for all $x, y \in \mathcal{R}$. The derivation $d$ is uniquely determined by $\mathfrak{F}$ and is called an associated derivation of $\mathfrak{F}$.

Very recently, Ali and Dur [2] initiated the study of \textsc{*-centralizing derivations in prime rings with involution and proved the \textsc{*-version of classical result due to Posner} [9]. Precisely, they proved that: let $\mathcal{R}$ be a prime ring with involution $'*$' such that $\text{char}(\mathcal{R}) \neq 2$. Let $d$ be a nonzero derivation on $R$ such that $[d(x), x^*] \in \mathcal{Z}(\mathcal{R})$ for all $x \in \mathcal{R}$. Then $\mathcal{R}$ is commutative. Latter this result was extended by Najjer et al. [8] for \textsc{*-centralizing derivations}. Recently, Alahmadi et al. [3], generalized above result as follows: let $\mathcal{R}$ be a prime ring with involution of the second kind such that $\text{char}(\mathcal{R}) \neq 2$. If $\mathcal{R}$ admits a nonzero generalized derivation $\mathfrak{F} : \mathcal{R} \to \mathcal{R}$ such that $[\mathfrak{F}(x), x^*] \in \mathcal{Z}(\mathcal{R})$ for all $x \in \mathcal{R}$, then $\mathcal{R}$ is commutative.

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Motivated by the above mentioned results studied in [2, 3, 8], we continue our line of investigation in the same direction for generalized derivation equipped with on skew Lie product. Finally, we conclude our manuscript with an example in support of our hypothesis of second kind involution, which shows that second kind assumption is an essential condition in our results.

2. THE RESULTS

We begin our discussion with the following known results.

Lemma 2.1. [1, Lemma 4] Let $\mathcal{R}$ be a prime ring with involution $^{\ast}$ of the second kind such that $\text{char} (\mathcal{R}) \neq 2$. If $\nabla [x, x^\ast] \in \mathcal{L} (\mathcal{R})$ for all $x \in \mathcal{R}$, then $\mathcal{R}$ is commutative.

Lemma 2.2. [8, Fact 1] Let $(\mathcal{R}, ^{\ast})$ be a 2-torsion free prime ring with involution $^{\ast}$ provided with a derivation $d$. Then $d(h) = 0$ for all $h \in H (\mathcal{R}) \cap \mathcal{L} (\mathcal{R})$ implies that $d(z) = 0$ for all $z \in \mathcal{L} (\mathcal{R})$.

The first main result of the present paper is the following theorem.

Theorem 2.1. Let $\mathcal{R}$ be a prime ring with involution $^{\ast}$ of the second kind such that $\text{char}(R) \neq 2$. Next, let $\mathcal{F}$ be a generalized derivation on $R$ associated with a nonzero derivation $d$ on $\mathcal{R}$ such that $\nabla [x, \mathcal{F}(x)] \pm x \circ x^\ast \in \mathcal{L} (\mathcal{R})$ for all $x \in \mathcal{R}$. Then $\mathcal{R}$ is commutative.

Proof. By the assumption, we have

$$(2.1) \quad \nabla [x, \mathcal{F}(x)] \pm x \circ x^\ast \in \mathcal{L} (\mathcal{R}) \quad \text{for all} \quad x \in \mathcal{R}. $$

This can be further re-written as

$$(2.2) \quad x\mathcal{F}(x) - \mathcal{F}(x^\ast) \pm x \circ x^\ast \in \mathcal{L} (\mathcal{R}) \quad \text{for all} \quad x \in \mathcal{R}. $$

Linearization of (2.2) yields that

$$(2.3) \quad x\mathcal{F}(y) + y\mathcal{F}(x) - \mathcal{F}(x)y^\ast - \mathcal{F}(y)x^\ast \pm x \circ y^\ast \pm y \circ x^\ast \in \mathcal{L} (\mathcal{R}) \quad \text{for all} \quad x, y \in \mathcal{R}. $$

Replacing $y$ by $yh_0$ in (2.3) where $h_0 \in H (\mathcal{R}) \cap \mathcal{L} (\mathcal{R})$, we obtain

$$x\mathcal{F}(y)h_0 + xyd(h_0) + y\mathcal{F}(x)h_0 - \mathcal{F}(x)y^\ast h_0 - \mathcal{F}(y)x^\ast h_0 - yx^\ast d(h_0) \pm $$

$$(2.4) \quad (x \circ y^\ast)h_0 \pm (y \circ x^\ast)h_0 \in \mathcal{L} (\mathcal{R}) \quad \text{for all} \quad x, y \in \mathcal{R} \quad \text{and} \quad h_0 \in H (\mathcal{R}) \cap \mathcal{L} (\mathcal{R}). $$

Combining (2.3) and (2.4), we get

$$(2.5) \quad (xy - yx^\ast)d(h_0) \in \mathcal{L} (\mathcal{R}) \quad \text{for all} \quad x, y \in \mathcal{R} \quad \text{and} \quad h_0 \in H (\mathcal{R}) \cap \mathcal{L} (\mathcal{R}). $$

Applying primeness of $\mathcal{R}$, implies that either $\nabla [x, y] \in \mathcal{L} (\mathcal{R})$ for all $x, y \in \mathcal{R}$ or $d(h_0) = 0$ for all $h_0 \in H (\mathcal{R}) \cap \mathcal{L} (\mathcal{R})$. Consider $\nabla [x, y] \in \mathcal{L} (\mathcal{R})$ for all $x, y \in \mathcal{R}$. Taking $y$ by $x^\ast$, then in view of Lemma 2.1, we obtain $\mathcal{R}$ is commutative. Now consider $d(h_0) = 0$ for all $h_0 \in H (\mathcal{R}) \cap \mathcal{L} (\mathcal{R})$, thus by Lemma 2.2
we obtain \( d(z) = 0 \) for all \( z \in \mathcal{L}(R) \). Replacing \( y \) by \( yk_0 \) in (2.3) and using \( d(z) = 0 \) for all \( z \in \mathcal{L}(R) \), we obtain

\[
(2.6) \quad x\mathcal{G}(y)k_0 + y\mathcal{G}(x)k_0 + \mathcal{G}(x)y^*k_0 - \mathcal{G}(y)x^*k_0 \mp (x \circ y^*)k_0 \pm (y \circ x^*)k_0 \in \mathcal{L}(R)
\]

for all \( x, y \in R \) and \( k_0 \in S(R) \cap \mathcal{L}(R) \). Combining (2.3) and (2.6) yields that \( 2(\mathcal{G}(x)y^* \mp x \circ y^*)k_0 \in \mathcal{L}(R) \) for all \( x, y \in R \) and \( k_0 \in H(R) \cap \mathcal{L}(R) \). Since \( \text{char}(R) \neq 2 \) and \( S(R) \cap \mathcal{L}(R) \neq (0) \), then we have \( \mathcal{G}(x)y^* \mp x \circ y^* \in \mathcal{L}(R) \) for all \( x, y \in R \). Replacing \( y \) by \( h_0 \) where \( h_0 \in H(R) \cap \mathcal{L}(R) \), we obtain \((\mathcal{G}(x) \mp 2x)h_0 \in \mathcal{L}(R) \) for all \( x \in R \) and \( h_0 \in H(R) \cap \mathcal{L}(R) \). Since \( H(R) \cap \mathcal{L}(R) \neq (0) \), then by the primeness of \( R \), we obtain \( \mathcal{G}(x) \mp 2x \in \mathcal{L}(R) \) for all \( x \in R \). This can be further written as \([\mathcal{G}(x), x] = 0\) for all \( x \in R \). Thus in view of [7, Theorem 3.1], we get \( R \) is commutative. This completes the proof of the theorem. \( \square \)

**Theorem 2.2.** Let \( R \) be a prime ring with involution ‘\( \ast \)’ of the second kind such that \( \text{char}(R) \neq 2 \). Next, let \( \mathcal{G} \) be a generalized derivation on \( R \) associated with a nonzero derivation \( d \) on \( R \) such that \( \mathcal{G}(\nabla[x,x^\ast]) \pm \nabla[x,x^\ast] \in \mathcal{L}(R) \) for all \( x \in R \). Then \( R \) is commutative.

**Proof.** By the given assumption, we have

\[
(2.7) \quad \mathcal{G}(\nabla[x,x^\ast]) \pm \nabla[x,x^\ast] \in \mathcal{L}(R) \quad \text{for all} \quad x \in R.
\]

This can be further written as

\[
(2.8) \quad \mathcal{G}(xx^\ast) - \mathcal{G}(x^2) \pm xx^\ast \mp x^2 \in \mathcal{L}(R) \quad \text{for all} \quad x \in R.
\]

Replacing \( x \) by \( xk_0 \) in (2.8) where \( k_0 \in S(R) \cap \mathcal{L}(R) \), we get

\[
(2.9) \quad -\mathcal{G}(xx^\ast)k_0^2 - xx^\ast d(k_0^2) - \mathcal{G}(x^2)k_0^2 - x^2 d(k_0^2) \mp xx^\ast k_0^2 \mp x^2 k_0^2 \in \mathcal{L}(R).
\]

for all \( x \in R \) and \( k_0 \in S(R) \cap \mathcal{L}(R) \). Combining (2.8) and (2.9), we obtain

\[
(2.10) \quad -xx^\ast d(k_0^2) - 2\mathcal{G}(x^2)k_0^2 - x^2 d(k_0^2) \mp 2x^2 k_0^2 \in \mathcal{L}(R)
\]

for all \( x \in R \) and \( k_0 \in S(R) \cap \mathcal{L}(R) \). Substitute \( xk_0 \) for \( x \) in (2.10) where \( k_0 \in S(R) \cap \mathcal{L}(R) \).

\[
(2.11) \quad xx^\ast d(k_0^2)k_0^2 - 2\mathcal{G}(x^2)k_0^4 - 2x^2 d(k_0^2)k_0^2 - x^2 d(k_0^2)k_0^2 \mp 2x^2 k_0^4 \in \mathcal{L}(R)
\]

for all \( x \in R \) and \( k_0 \in S(R) \cap \mathcal{L}(R) \). Combining (2.10) ans (2.11), we get

\[
-2xx^\ast d(k_0^2)k_0^2 + 2x^2 d(k_0^2)k_0^2 \in \mathcal{L}(R) \quad \text{for all} \quad x \in R \quad \text{and} \quad k_0 \in S(R) \cap \mathcal{L}(R).
\]

This implies that

\[
-2(\nabla[x,x^\ast])d(k_0^2)k_0^2 \in \mathcal{L}(R) \quad \text{for all} \quad x \in R \quad \text{and} \quad k_0 \in S(R) \cap \mathcal{L}(R).
\]

Since \( \text{char}(R) \neq 2 \) and \( S(R) \cap \mathcal{L}(R) \neq (0) \). This implies that

\[
-(\nabla[x,x^\ast])d(k_0^2) \in \mathcal{L}(R) \quad \text{for all} \quad x \in R \quad \text{and} \quad k_0 \in S(R) \cap \mathcal{L}(R).
\]
By the primeness of \( \mathcal{R} \), we have either \( \mathcal{V}[x,x'] \in \mathcal{L}(\mathcal{R}) \) for all \( x \in \mathcal{R} \) and \( d(k_0^2) = 0 \) for all \( k_0 \in S(\mathcal{R}) \cap \mathcal{L}(\mathcal{R}) \). If we consider \( \mathcal{V}[x,x'] \in \mathcal{L}(\mathcal{R}) \) for all \( x \in \mathcal{R} \). Therefore in view of Lemma 2.1, we get \( \mathcal{R} \) is commutative. Now we consider

\[
d(k_0^2) = 0 \text{ for all } k_0 \in S(\mathcal{R}) \cap \mathcal{L}(\mathcal{R}).
\]

Replacing \( k_0 \) by \( k_0 + k' \) in (2.12) where \( k_0, k' \in S(\mathcal{R}) \cap \mathcal{L}(\mathcal{R}) \) and using (2.12), we obtain

\[
2d(k_0k') = 0 \text{ for all } k_0, k' \in S(\mathcal{R}) \cap \mathcal{L}(\mathcal{R}).
\]

Since \( \text{char}(\mathcal{R}) \neq 2 \), this implies that

\[
d(k_0k') = 0 \text{ for all } k_0, k' \in S(\mathcal{R}) \cap \mathcal{L}(\mathcal{R}).
\]

Replacing \( k' \) by \( k'h_0 \) in (2.13) where \( h_0 \in H(\mathcal{R}) \cap \mathcal{L}(\mathcal{R}) \) and using (2.13) with \( S(\mathcal{R}) \cap \mathcal{L}(\mathcal{R}) \neq (0) \), we get \( d(h_0) = 0 \) for all \( h_0 \in H(\mathcal{R}) \cap \mathcal{L}(\mathcal{R}) \). Therefore by Lemma 2.2, we get \( d(z) = 0 \) for all \( z \in \mathcal{L}(\mathcal{R}) \). Now linearization of (2.8), yields that

\[
\mathcal{F}(xy^s) + \mathcal{F}(yx^s) - \mathcal{F}(x'y^s) - \mathcal{F}(y'x^s) = \pm xy^s \pm yx^s \mp x'y' \mp y'x^s \in \mathcal{L}(\mathcal{R}).
\]

for all \( x, y \in \mathcal{R} \). Replacing \( y \) by \( yk_0 \) in (2.14) where \( k_0 \in S(\mathcal{R}) \cap \mathcal{L}(\mathcal{R}) \) and \( d(z) = 0 \) for \( z \in \mathcal{L}(\mathcal{R}) \), we obtain

\[
\mp xy^s k_0 = \mathcal{F}(xy^s)k_0 + \mathcal{F}(x'y^s)k_0 + \mathcal{F}(y'x^s)k_0 \mp xy^s k_0 \pm yx^s k_0
\]

(2.15)

\[
\pm xy^s k_0 = \mathcal{F}(xy^s)k_0 \in \mathcal{L}(\mathcal{R}) \text{ for all } x, y \in \mathcal{R} \text{ and } k_0 \in S(\mathcal{R}) \cap \mathcal{L}(\mathcal{R}).
\]

Combining (2.14) and (2.15), we obtain

\[
2(\mathcal{F}(yx^s) \pm yx^s) k_0 \in \mathcal{L}(\mathcal{R}) \text{ for all } x, y \in \mathcal{R} \text{ and } k_0 \in S(\mathcal{R}) \cap \mathcal{L}(\mathcal{R}).
\]

Since \( \text{char}(\mathcal{R}) \neq 2 \) and \( S(\mathcal{R}) \cap \mathcal{L}(\mathcal{R}) \neq (0) \), this implies that

\[
\mathcal{F}(yx^s) = \pm yx^s \in \mathcal{L}(\mathcal{R}) \text{ for all } x, y \in \mathcal{R}.
\]

Again taking \( h_0 \) for \( x \) in (2.16), and using \( d(z) = 0 \) for all \( z \in \mathcal{L}(\mathcal{R}) \), \( S(\mathcal{R}) \cap \mathcal{L}(\mathcal{R}) \neq (0) \), we obtain

\[
\mathcal{F}(y) = \pm y \in \mathcal{L}(\mathcal{R}) \text{ for all } x, y \in \mathcal{R}.
\]

This can be further written as \([\mathcal{F}(y), y] = 0 \) for all \( y \in \mathcal{R} \). Therefore in view of [7, Theorem 3.1], we get the required result. This complete the proof of theorem. \( \square \)

**Theorem 2.3.** Let \( \mathcal{R} \) be a prime ring with involution \( \ast \) of the second kind such that \( \text{char}(\mathcal{R}) \neq 2 \). Next, let \( \mathcal{F} \) be a generalized derivation on \( \mathcal{R} \) associated with a nonzero derivation \( d \) on \( \mathcal{R} \) such that \( \mathcal{F}(\mathcal{V}[x,x']) = \pm x'x^s \in \mathcal{L}(\mathcal{R}) \) for all \( x \in \mathcal{R} \) Then \( \mathcal{R} \) is commutative.
Proof. We have
\[ \mathfrak{g}(\nabla[x,x^*]) \pm x \circ x^* \in \mathcal{L}(R) \text{ for all } x \in R. \]
This implies that
\[ (2.18) \quad \mathfrak{g}(xx^*) - \mathfrak{g}(x^2) \pm x \circ x^* \in \mathcal{L}(R) \text{ for all } x \in R. \]
Substituting \( xk_0 \) for \( x \) in (2.18) where \( k_0 \in S(R) \cap \mathcal{L}(R) \neq (0) \), we get
\[ (2.19) \quad -\mathfrak{g}(xx^*)k_0^2 - xx^*d(k_0^2) - \mathfrak{g}(x^2)k_0^2 - x^2d(k_0^2) \pm (x \circ x^*)k_0^2 \in \mathcal{L}(R) \]
for all \( x \in R \) and \( k_0 \in S(R) \cap \mathcal{L}(R) \). Combining (2.18) and (2.19), we obtain
\[ (2.20) \quad -xx^*d(k_0^2) - 2\mathfrak{g}(x^2)k_0^2 - x^2d(k_0^2) \in \mathcal{L}(R) \text{ for all } x \in R. \]
and \( k_0 \in S(R) \cap \mathcal{L}(R) \). Replacing \( x \) by \( xk_0 \) in (2.20) where \( k_0 \in S(R) \cap \mathcal{L}(R) \).
\[ (2.21) \quad xx^*d(k_0^2) - 2\mathfrak{g}(x^2)k_0^2 - 2x^2d(k_0^2)k_0^2 - x^2d(k_0^2) \in \mathcal{L}(R) \]
for all \( x \in R \) and \( k_0 \in S(R) \cap \mathcal{L}(R) \). Combining (2.20) and (2.21), we get
\[ (2.22) \quad 2(xx^* - x^2)d(k_0^2)k_0^2 \in \mathcal{L}(R) \text{ for all } x \in R \text{ and } k_0 \in S(R) \cap \mathcal{L}(R). \]
Since \( \text{char}(R) \neq 2 \) and \( S(R) \cap \mathcal{L}(R) \neq (0) \), this implies that
\[ (2.23) \quad \nabla[x,x^*]d(k_0^2) \in \mathcal{L}(R) \text{ for all } x \in R \text{ and } k_0 \in S(R) \cap \mathcal{L}(R). \]
Then by the primeness of \( R \), in view of Lemma 2.1 \( R \) is commutative or \( d(k_0^2) = 0 \) for all \( k_0 \in S(R) \cap \mathcal{L}(R) \), which is same as (2.12). Now follow the same steps as we did after (2.12), we get required result. This complete the proof of theorem. \( \square \)

Corollary 2.4. Let \( R \) be a prime ring with involution \( ^* \) of the second kind such that \( \text{char}(R) \neq 2 \).
Next, let \( \delta \) be a nonzero derivation on \( R \) such that \( \delta(\nabla[x,x^*]) \pm x \circ x^* \in \mathcal{L}(R) \) for all \( x \in R \). Then \( R \) is commutative.

The following example shows that the second kind involution is an essential condition in Theorem 2.1.

Example 2.3. Let \( R = \left\{ \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix} \begin{array}{ll} \beta_1, \beta_2, \beta_3, \beta_4 \in \mathbb{Z} \end{array} \right\} \).
Of course, \( R \) with matrix addition and matrix multiplication is a prime ring. Define mappings
\( \mathfrak{g} \), \( \delta^* : R \rightarrow R \) such that
\[ \mathfrak{g} \left( \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix} \right) = \begin{pmatrix} 0 & -\beta_2 \\ \beta_3 & 0 \end{pmatrix}, \]
\[ \delta \left( \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix} \right) = \begin{pmatrix} 0 & -\beta_2 \\ \beta_3 & 0 \end{pmatrix}, \]
\[ \left( \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_3 & \beta_4 \end{pmatrix} \right)^* = \begin{pmatrix} \beta_4 & -\beta_2 \\ -\beta_3 & \beta_1 \end{pmatrix}. \]
Obviously, \( \mathcal{L}(R) = \left\{ \begin{pmatrix} \beta_1 & 0 \\ 0 & \beta_1 \end{pmatrix} \begin{array}{l} \beta_1 \in \mathbb{Z} \end{array} \right\} \).
Then \( x^* = x \) for all \( x \in \mathcal{L}(R) \), and hence \( \mathcal{L}(R) \subseteq \]
which shows that the involution \( ' \alpha ' \) is of the first kind. Moreover, \( \xi, d \) are nonzero generalized derivation and an associated derivation defined as above, such that the identity of Theorem 2.1 satisfied, but \( R \) is not commutative. Hence, the hypothesis of the second kind involution is crucial in our results.

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