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INTRINSIC CHARACTERIZATION OF C^* (UB)- SEGAL ALGEBRAS; AND HARMONIC ANALYSIS ON WEIGHTED ABELIAN SEMIGROUPS

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1. Tribute

I am very fortunate to have Professor Subhash Bhatt as my Ph.D. guide and I am trained under his able headship in the Department of Mathematics, Sardar Patel University. Professor Bhatt has played a vital role in moulding my mathematical career. I heard his name when I was in M.Sc. from the faculty members in the Department of Mathematics, Gujarat University. In 2004, when I had gone to Vallabh Vidyanagar to appear for NBHM M.Sc. scholarship test, I had met him first and I expressed my wish to do Ph.D. under his guidance. He immediately replied that "pahela NET pass kar ane pachhi aavje". I remember the day when I went there for admission in Ph.D. under the guidance of both Professor Subhash Bhatt and Professor H. V. Dedania; he insisted that I should have lunch at his home and I had lunch with him at his home. He had also visited my home on the very first day when I shifted to Vallabh Vidyanagar from Ahmedabad.

I have never seen him running after money but have seen him running hard after research. He used to be there in office even after office hours. Many a times after our discussions or whenever some new idea came to his mind he used to telephone and used to say "I called you to share, otherwise I may forget". He also told me "never go to meeting without preparation". There are number of occasions from which I learned Mathematics or life lessons from him. I met him twice in a day in our last meeting. Till today, he is remembered and missed naturally in sharing my joy of solving research problems or whenever I get stuck somewhere in solving problems or to discuss new ideas.

He has more than 100 research papers to his credit which are published in high quality national and international journals. He is internationally very well known. Unfortunately, he was not as popular as many who are popular by solving some school level problems or by giving popular talks. I have seen his vision, passion, analysis, way of working, critical thinking, accuracy, dedication and have learnt a lot. The nation and in particular Gujarat have lost a finest mathematician.

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2. Our Research

My work with Professor Bhatt is in two areas namely Banach algebras and Harmonic analysis on weighted (semi)groups.

2.1. **Banach algebras.** There has been considerable interest in Banach algebras that are dense ideals in Banach algebras (respectively C^*- algebras), called *Segal algebras* (respectively *C^*-Segal algebras*). A commutative C^*- Segal algebra incorporates Nachbin's weighted function algebras in its Gelfand-Naimark framework [1]; and at the non-commutative level, this leads to weighted C^*-algebras (a non-commutative analogue of Nachbin algebras) [15]. A faithful Banach algebra $\mathcal{A}$ is a *C^*- Segal algebra* if $\mathcal{A}$ continuously sits in a C^*-algebra $\mathcal{B}$ as a dense ideal under an injective homomorphism i. If $i(\mathcal{A})$ is closed under the involution of $\mathcal{A}$, then $\mathcal{A}$ is *self-adjoint.* In this case, $\mathcal{A}$ becomes a Banach $*$-algebra. Let $\mathcal{A}$ be a Banach $*$- algebra, and let $s_{\mathcal{A}}(a) = \sqrt{r(a^*a)}$ $(a \in \mathcal{A})$ be the Pták function, where $r(\cdot)$ is the spectral radius [7]. Let $\mathcal{A}$ be a Banach algebra. A pair $m = (m_l, m_r)$ of bounded linear maps from $\mathcal{A}$ to $\mathcal{A}$ satisfying $m_l(ab) = m_l(a)b$, $m_r(ab) = am_r(b)$ and $am_l(b) = m_r(a)b$ for all $a, b \in \mathcal{A}$ is a *multiplier* on $\mathcal{A}$. Let $M(\mathcal{A})$ be the collection of all multipliers on $\mathcal{A}$. Then $M(\mathcal{A})$ is a Banach algebra with the norm $\|m\| = \max\{\|m_l\|_{op}, \|m_r\|_{op}\}$ $(m = (m_l, m_r) \in M(\mathcal{A}))$, where $\|\cdot\|_{op}$ is the operator norm. Further, if $\mathcal{A}$ is a Banach $*$- algebra, then $M(\mathcal{A})$ is a Banach $*$-algebra with involution $m \mapsto m^* = (m_r^*, m_l^*)$, $m_l^*(x) = m_l(x^*)^*$, $m_r^*(x) = m_r(x^*)^*$ for all $x \in \mathcal{A}$. The following is an intrinsic characterization of self-adjoint C^*-Segal algebras.

Theorem 2.1 ((7), Theorem 1)**.** *Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a faithful Banach $*$-algebra. Then the following are equivalent:*

(1) *$\mathcal{A}$ is a self-adjoint C^*-Segal algebra.*
(2) *The Banach $*$-algebra $M(\mathcal{A})$ contains a C^*-subalgebra which is a C^*-algebra containing $\mathcal{A}$, and there is $l > 0$ such that $\|ab\|_{\mathcal{A}} \leq l s_{\mathcal{A}}(a)\|b\|_{\mathcal{A}}$ and $\|ab\|_{\mathcal{A}} \leq l\|a\|_{\mathcal{A}} s_{\mathcal{A}}(b)$ for all $a, b \in \mathcal{A}$.*
(3) *$\mathcal{A}$ is hermitian, $*$-semisimple and there is $l > 0$ such that $\|ab\|_{\mathcal{A}} \leq l\|a\|_{\mathcal{A}} s_{\mathcal{A}}(b)$ and $\|ab\|_{\mathcal{A}} \leq l s_{\mathcal{A}}(a)\|b\|_{\mathcal{A}}$ for all $a, b \in \mathcal{A}$.*

Corollary 2.2 ((7), Corollary 2)**.** *Let $\mathcal{A}$ be a self-adjoint C^*- Segal algebra in a C^*- algebra $\mathcal{B}$. Then B is unique; $\mathcal{B}$ is the C^*-algebra $C^*(\mathcal{A})$ of $\mathcal{A}$; and $\mathcal{A}$ is a dense ideal in $C^*(\mathcal{A})$.*

Thus, if $\mathcal{A}$ is a self-adjoint C^*- Segal algebra in a C^*- algebra $\mathcal{B}$, then $\mathcal{B}$ is necessarily unique and is the enveloping C^*- algebra $C^*(\mathcal{A})$ of $\mathcal{A}$.

Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a Banach algebra. A norm $\|\cdot\|$ is a *Segal norm* on $\mathcal{A}$ if there exist $k, l > 0$ such that $k\|a\|_M \leq |a| \leq l\|a\|_{\mathcal{A}}$ $(a \in \mathcal{A})$, where $\|a\|_M = \sup\{\|ab\|_{\mathcal{A}}, \|ba\|_{\mathcal{A}} : b \in \mathcal{A}, \|b\|_{\mathcal{A}} \leq 1\}$ is the *multiplier seminorm* on $\mathcal{A}$. A multiplier seminorm $\|\cdot\|_M$ is a norm on $\mathcal{A}$ if $\mathcal{A}$ is faithful, i.e., if $a \in \mathcal{A}\backslash\{0\}$, then there are $m, n \in \mathcal{A}$ such that $am \neq 0$ and $na \neq 0$. By [15], a Banach algebra $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a *Segal extension* of $\mathcal{A}$ if and only if $\mathcal{B}$ is the completion of $\mathcal{A}$ with respect to a *Segal norm.* We considered the case when $\mathcal{A}$ is not necessarily faithful. This forces us to consider Segal seminorms and Segal extensions not necessarily injective.

Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a Banach algebra, and let p be an algebra seminorm on $\mathcal{A}$. Let $I = \{a \in \mathcal{A} : p(a) = 0\}$. Then the completion of $\mathcal{A}/I$ with respect to the norm $\widetilde{p}(x + I) = p(x)$ $(x \in \mathcal{A})$ is the *Hausdorff completion* of $\mathcal{A}$ with respect to p.

Definition 2.3. *An algebra seminorm p on a Banach algebra $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is a* Segal seminorm of type I *if there are positive constants k, l such that $k\|a\|_M \leq p(a) \leq l\|a\|_{\mathcal{A}}$ $(a \in \mathcal{A})$; and p is* Segal seminorm of type II *if the induced norm $\widetilde{p}(a + I) = p(a)$ $(a \in \mathcal{A})$ on the quotient algebra $\mathcal{A}/I$ is a Segal norm, where $I = \{a \in \mathcal{A} : p(a) = 0\}$.*

Notice that a Segal seminorm of type I is of type II.

Definition 2.4. *A Segal extension of a Banach algebra $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ is a pair $(\mathcal{B}, \varphi)$, where*

(1) *$(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a Banach algebra;*
(2) *$\varphi : \mathcal{A} \to \mathcal{B}$ is a continuous algebra homomorphism, not necessarily injective;*
(3) *$\varphi(\mathcal{A})$ is a dense ideal in $\mathcal{B}$.*

Proposition 2.5 ((8), Proposition 1)**.** *The following conditions are equivalent for a Banach algebra $(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$.*

(1) *$(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is a Segal extension of $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$.*
(2) *$(\mathcal{B}, \|\cdot\|_{\mathcal{B}})$ is the Hausdorff completion of $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ with respect to a Segal seminorm of type II on $\mathcal{A}$.*

Theorem 2.6 ((8), Theorem 1)**.** *Let $(\mathcal{A}, \|\cdot\|)$ be a Banach algebra. Let $I = \{a \in \mathcal{A} : \|a\|_M = 0\}$. Then the following hold.*

(1) *The Hausdorff completion $(\mathcal{A}/I, \|\cdot\|_{M}^{\sim})^{\sim}$ of $\mathcal{A}$ in the multiplier seminorm $\|\cdot\|_M$ is a Segal extension of $\mathcal{A}$.*
(2) *If $\mathcal{B}$ is a Segal extension of $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ defined by a Segal seminorm p of type I on $\mathcal{A}$, then there exists an algebra homomorphism from $\mathcal{B}$ to $(\mathcal{A}/I, \|\cdot\|_{M}^{\sim})^{\sim}$ with dense range.*

The Gelfand map on a commutative Banach algebra, which is not necessarily semisimple, and the Gelfand-Naimark-Segal construction on a not necessarily $*$- semisimple and not necessarily commutative Banach $*$-algebra are two natural constructs that are non-injective homomorphisms. When are these homomorphims Segal extensions? When are the spectral radius (in commutative case) and Gelfand-Naimark-Segal pseudo norm (in not necessarily commutative, involutive case) Segal seminorms? The following answers these.

Corollary 2.7 ((8), Corollary 1)**.** *Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a commutative Banach algebra. The Gelfand map $\Phi : \mathcal{A} \to \widehat{\mathcal{A}}$, $\Phi(a) = \widehat{a}$, is a Segal extension of $\mathcal{A}$ if and only if the spectral radius is a Segal seminorm of type II if and only if there is $k > 0$ such that for all $a, b \in \mathcal{A}$,*

$$k\|ab + rad\mathcal{A}\|_{\mathcal{A}/rad\mathcal{A}} \leq r(a)\|b + rad\mathcal{A}\|_{\mathcal{A}/rad\mathcal{A}}.$$

In particular, this holds if there is $k > 0$ such that $k\|ab\|_{\mathcal{A}} \leq r(a)\|b\|_{\mathcal{A}}$ for all $a, b \in \mathcal{A}$.

Corollary 2.8 ((8), Corollary 2)**.** *Let $(\mathcal{A}, \|\cdot\|)$ be a Banach $*$- algebra.*

(1) *Let f be a continuous positive linear functional on $\mathcal{A}$, and let (π_f, H_f) be the GNS representation defined by f. Let $C^*(\pi_f(\mathcal{A}))$ be the C^*- algebra on H_f generated by $\pi_f(\mathcal{A})$. Let $I = \{x \in \mathcal{A} : f(y^*x^*xy) = 0 \ (y \in \mathcal{A})\}$. Then $(\pi_f, C^*(\pi_f(\mathcal{A})))$ is a Segal extension of $\mathcal{A}$ if and only if there is $k > 0$ such that*

$$k\|x + I\|_M \leq \sup\{\frac{f(y^*x^*xy)^{\frac{1}{2}}}{f(y^*y)^{\frac{1}{2}}} : f(y^*y) \neq 0\}.$$

In particular, this holds if there is $k > 0$ such that

$$k\|x\|_M \leq \sup\{\frac{f(y^*x^*xy)^{\frac{1}{2}}}{f(y^*y)^{\frac{1}{2}}} : f(y^*y) \neq 0\}.$$

(2) *The map j is a Segal extension if and only if there are $k, l > 0$ such that for all $a, b \in \mathcal{A}$, $k \max\{\|ab + srad\mathcal{A}\|_{\mathcal{A}/srad\mathcal{A}}, \|ba + srad\mathcal{A}\|_{\mathcal{A}/srad\mathcal{A}}\} \leq p_\infty(a)\|b + srad\mathcal{A}\|_{\mathcal{A}/srad\mathcal{A}}$. This, in particular, holds if there is $k > 0$ such that $k \max\{\|ab\|, \|ba\|\} \leq p_\infty(a)\|b\|$ for all $a, b \in \mathcal{A}$.*

A Banach algebra is a *uB-Segal algebra* if it has a Segal extension $(\mathcal{B}, i)$, where $\mathcal{B}$ is a uniform Banach algebra and $i : \mathcal{A} \to \mathcal{B}$ is a continuous injective homomorphism such that $i(\mathcal{A})$ is a dense ideal in $\mathcal{B}$. Thus a uB-Segal algebra is a Banach algebra that is a Segal algebra in a uB-algebra. If $\mathcal{A}$ is a uB-Segal algebra, then $\mathcal{A}$ is necessarily commutative. We call such an i to be a uB-Segal extension. The following describes a canonical uB-Segal algebra associated with a semisimple commutative Banach algebra.

Theorem 2.9 ((8), Theorem 2)**.** *Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a semisimple commutative Banach algebra with a bounded approximate identity. Let $\mathcal{A}_s$ be the collection of all those $x \in \mathcal{A}$ such that $\psi x \in \mathcal{A}$ for all $\psi \in U(A)$ and $\|x\|_s = \sup\{\frac{\|\psi x\|}{\|\psi\|_\infty} : \psi \in U(\mathcal{A}), \psi \neq 0\} < \infty$. Then the following hold.*

(1) *$(\mathcal{A}_s, \|\cdot\|_s)$ is a Banach subalgebra of $\mathcal{A}$ continuously embedded in $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$.*

(2) *$\mathcal{A}_s$ is uB-Segal algebra in $U(\mathcal{A})$.*

(3) *$\mathcal{A}_s$ is the largest Banach subalgebra of $\mathcal{A}$ that sits as a uB-Segal algebra in $U(\mathcal{A})$.*

The following is a uniform algebra analogue of a result on C^*-Segal algebras in (7).

Theorem 2.10 ((8), Theorem 3)**.** *Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a faithful commutative Banach algebra. Then the following are equivalent.*

(i) *$\mathcal{A}$ is a uB-Segal algebra.*

(ii) *The Banach algebra $M(\mathcal{A})$ contains a subalgebra that is a uB-algebra containing $\mathcal{A}$ and there is $l > 0$ such that $\max\{\|ab\|_{\mathcal{A}}, \|ba\|_{\mathcal{A}}\} \leq l r_{\mathcal{A}}(a)\|b\|_{\mathcal{A}}$ for all $a, b \in \mathcal{A}$.*

(iii) *$\mathcal{A}$ is semisimple and there is $l > 0$ such that $\max\{\|ab\|_{\mathcal{A}}, \|ba\|_{\mathcal{A}}\} \leq l r_{\mathcal{A}}(a)\|b\|_{\mathcal{A}}$ for all $a, b \in \mathcal{A}$.*

Like C^*- Segal algebras, for uB- Segal algebras we have

Corollary 2.11 ((8), Corollary 3)**.** *Let $(\mathcal{A}, \|\cdot\|_{\mathcal{A}})$ be a uB-Segal algebra in a uniform Banach algebra $\mathcal{B}$. Then $\mathcal{B}$ is unique, $\mathcal{B}$ is the uB-algebra $U(\mathcal{A})$ of $\mathcal{A}$, and $\mathcal{A}$ is dense ideal in $U(\mathcal{A})$.*

Thus, if $\mathcal{A}$ is a uB- Segal algebra in a uniform Banach algebra $\mathcal{B}$, then $\mathcal{B}$ is necessarily unique and is the enveloping uB- algebra $U(\mathcal{A})$ of $\mathcal{A}$.

2.2. **Harmonic analysis on weighted (semi)groups.** Let S be an abelian semigroup. A map $\omega : S \to (0, \infty)$ is a *weight* if $\omega(st) \leq \omega(s)\omega(t)$ $(s, t \in S)$. If ω is a weight on a semigroup S, then (S, ω) is a weighted semigroup. A weight on a semigroup is an analogues of norm on an algebra. Presumably, the number $\omega(s)$ is a size or frequency of s in S. A map $\alpha : S \to S$ is a multiplier if $\alpha(st) = s\alpha t = \alpha(s)t$ $(s, t \in S)$. Let $M(S)$ be the set of all multipliers on S. Then $M(S)$ is a unital semigroup with composition operation. A multiplier α on S is ω-bounded if there is $K > 0$ such that $\omega(\alpha s) \leq K\omega(s)$ $(s \in S)$. Let $M_\omega(S)$ be the set of all ω-bounded multipliers on S. A semigroup S is faithful if $s = t$ whenever $s, t \in S$ and $su = tu$ for all $u \in S$. If S is a faithful, then S is embedded in $M(S)$ and $M_\omega(S)$ via the map $s \mapsto \gamma_s$, where $\gamma_s(t) = st$ $(t \in S)$. In fact, S is a semigroup ideal in both $M(S)$ and $M_\omega(S)$. Define $\widetilde{\omega}$ on $M_\omega(S)$ by

$$\widetilde{\omega}(\alpha) = \sup\{\frac{\omega(\alpha s)}{\omega(s)} : s \in S\} \quad (\alpha \in M_\omega(S)).$$

Then $\widetilde{\omega}$ is a weight on $M_\omega(S)$ and so $(M_\omega(S), \widetilde{\omega})$ is a natural weighted semigroup associated with (S, ω). The weighted semigroup $(M_\omega(S), \widetilde{\omega})$ behaves like the multiplier algebra of a commutative Banach algebra.

Let (S, ω) be a weighted abelian semigroup. Let

$$\ell^1(S, \omega) = \{f : S \to \mathbb{C} : \|f\|_\omega = \sum_{s \in S} |f(s)|\omega(s) < \infty\}.$$

Then $\ell^1(S, \omega)$ is a commutative Banach algebra with the norm $\|\cdot\|_\omega$ and the convolution multiplication

$$(f \star g)(s) = \sum_{uv=s} f(u)g(v) \qquad (f, g \in \ell^1(S, \omega), s \in S).$$

The above convolution is considered to be 0 at $s \in S$ if $uv = s$ has no solution in S. The Banach algebra $\ell^1(S,\omega)$ called a *Beuring algebra.* The objective of study the weighted semigroup is to determine the properties of the Banach algebra $\ell^1(S,\omega)$ and viceversa. For example, by [12], $\ell^1(S)$ is semisimple if and only if S is separating, by [9], $\ell^1(S,\omega)$ is semisimple if and only if S is separating and ω is semisimple weight. So, properties of semigroup and weight play an important role in the structure of the associated Beurling algebra $\ell^1(S,\omega)$.

Let I be an ideal in an abelian semigroup S. Then the Rees quotient, S/I, of S by I is a semigroup consisting of I and s which are not in I. The semigroup operation on S/I is given by $s \cdot I = I$ for all $s \in S$, $I \cdot I = I$, $s \cdot t = I$ if $s \in I$ or $t \in I$ and $s \cdot t = st$ if $s \notin I$ and $t \notin I$. If ω is a weight on S with $\omega_0 = \inf\{\omega(s) : s \in S\} > 0$, the we define $\widetilde{\omega}_q(S) = 1$ and $\widetilde{\omega}_q(\alpha) = \widetilde{\omega}$ if $\alpha \notin S$. Then $\widetilde{\omega}_q$ is a weight on $M_\omega(S)/S$. So, given an abelian weighted semigroup (S,ω) we have two more abelian weighted semigroups associated with it namely $(M_\omega(S), \widetilde{\omega})$ and $(M_\omega(S)/S, \widetilde{\omega}_q)$. We shall see the interplay between these weighted semigroups and their Beurling algebras.

We shall deal with the following types of abelian semigroups and weights.

Definition 2.12. *Let S be an abelian semigroup, and let ω be a weight on S.*

(1) *S is* separating *if $s = t$ whenever $s, t \in S$ and $s^2 = t^2 = st$.*

(2) *S is* cancellative *if $s = t$ whenever $s, t \in S$ and $su = tu$ for some $u \in S$.*

(3) *S is an* inverse semigroup *if given s there is unique $t \in S$ such that $sts = s$ and $tst = t$.*

(4) *ω is* semisimple [9] *if $\lim_{n\to\infty} \omega(s^n)^{\frac{1}{n}} > 0$ for all $s \in S$.*

(5) *ω is* radical [9] *if $\lim_{n\to\infty} \omega(s^n)^{\frac{1}{n}} = 0$ for all $s \in S$.*

(6) *ω is a* Beurling-Domar weight [10] *if $\omega \geq 1$ and $\sum_{n\in\mathbb{N}} \frac{\log \omega(s^n)}{1+n^2} < \infty$ for all $s \in S$.*

(7) *ω is a* GRS- weight [11] *if $\lim_{n\to\infty} \omega(s^n)^{\frac{1}{n}} = 1$ for all $s \in S$.*

(8) *ω is a* uniform weight *if $\omega(s^2) = \omega(s)^2$ for all $s \in S$.*

(9) *Let S be involutive. Then ω is a* C^*- weight *if $\omega(s^*s) = \omega(s)^2$ for all $s \in S$.*

(10) *ω is* weakly regular *if $\widetilde{\omega}|_S$ and ω are equivalent, i.e., there exists $k > 0$ such that $k\omega(s) \leq \widetilde{\omega}(\gamma_s)$ for all $s \in S$.*

(11) *ω is a* regular weight *if $\widetilde{\omega} = \omega$ on S.*

The following gives some properties of these weighted semigroups.

Theorem 2.13 ((2))**.** *Let (S,ω) be a weighted abelian semigroup. Then the following hold.*

(1) *Let $\alpha : S \to S$ be a map such that $s\alpha t = (\alpha s)t$ $(s, t \in S)$. Then α is a multiplier.*

(2) *The set $M(S)$ is a unital abelian semigroup with composition; and S is embedded in $M(S)$ via $s \mapsto \gamma_s$ as an ideal of $M(S)$.*

(3) *For any weight ω on S, $M_\omega(S)$ is a subsemigroup of $M(S)$ and S is an ideal in $M_\omega(S)$.*

(4) *If S is involutive and ω is symmetric, then each of $M(S)$ and $M_\omega(S)$ are involutive and S is a $*$-ideal.*

(5) *$S = M(S)$ if and only if S is unital.*

(6) *If S has a finite set of relative units, then $M(S) = M_\omega(S)$ for all weight ω on S.*

(7) *There exists a weighted semigroup (S, ω) such that $M_\omega(S) \neq M(S)$.*

(8) *There exists a semigroup S such that $S_e \neq M(S)$, where S_e is the unitization of S.*

(9) *$\widetilde{\omega}$ a weight $M_\omega(S)$.*

(10) *S is an inverse semigroup if and only if $M_\omega(S)$ is an inverse semigroup.*

(11) *Let S be involutive and ω be symmetric. Then $\widetilde{\omega}$ is symmetric; and S is $*$-separating if and only if $M_\omega(S)$ is $*$-separating.*

(12) *S is separating if and only if $M_\omega(S)$ is separating.*

(13) *There exists a semigroup S such that both S and $M(S)$ are separating; but the quotient $M(S)/S$ fails to be separating.*

(14) *ω is semisimple on S if and only if $\widetilde{\omega}$ is semisimple on $M_\omega(S)$. If ω is a uniform weight or a C^*- weight on S, then ω is a semisimple weight on S.*

(15) *If $\widetilde{\omega}$ is a Beurling-Domar weight on $M_\omega(S)$, then ω is a Beurling-Domar weight on S.*

(16) *Let (S, ω) satisfy any of the following conditions:*

 (a) *For each $\alpha \in M_\omega(S)$, there exists $m \in \mathbb{N}$ such that $\alpha^m \in S$.*

 (b) *Every element of S is idempotent.*

 If ω is a Beurling-Domar weight, then $\widetilde{\omega}$ is a Beurling-Domar weight.

(17) *Let ω be semisimple. Then $\nu_\omega(s) := \lim_{n\to\infty} \omega(s^n)^{\frac{1}{n}}$ $(s \in S)$ is a uniform weight, and it is the largest uniform weight dominated by ω.*

(18) *Let ω be semisimple. Then $\mu_\omega(s) = \nu_\omega(s^*s)^{\frac{1}{2}}$ $(s \in S)$ is a C^*- weight, and it is the largest C^*- weight dominated by ω.*

The following is relation between the Beurling algebras on the weighted semigroups (S, ω), $(M_\omega(S), \widetilde{\omega})$ and $(M_\omega(S)/S, \widetilde{\omega}_q)$.

Theorem 2.14 ((2), Theorem 3.1). *Let ω be weakly regular with $\omega_0 > 0$. Then $\ell^1(S, \omega)$ is a closed ideal of $\ell^1(M_\omega(S), \widetilde{\omega})$ and the quotient algebra $\ell^1(M_\omega(S), \widetilde{\omega})/\ell^1(S, \omega)$ is isomorphic to the Beurling algebra of the Rees quotient semigroup $M_\omega(S)/S$ with the quotient weight $\widetilde{\omega}_q$.*

Let (S, ω) be a weighted abelian semigroup. A nonzero map $\chi : S \to \mathbb{C}$ is a *semicharacter* on S if $\chi(st) = \chi(s)\chi(t)$ $(s, t \in S)$. A semicharacter χ on S is ω- *bounded* if $|\chi(s)| \leq \omega(s)$ for all $s \in S$. Let $\Phi_{\omega s}(S)$ be the set of all ω- bounded semicharacters on S. The ω- bounded semicharacters on S are important as they are directly connected to complex homomorphisms on $\ell^1(S, \omega)$. In fact, given an ω- bounded semicharacter χ, define φ_χ on $\ell^1(S, \omega)$ by

$$\varphi_\chi(f) = \sum_{s \in S} f(s)\chi(s) \qquad (f \in \ell^1(S, \omega)).$$

Then φ_χ is a complex homomorphism on $\ell^1(S,\omega)$ and conversely, given a complex homomorphism φ on $\ell^1(S,\omega)$ there is an ω- bounded semicharater χ on S such that $\varphi = \varphi_\chi$. The following result giving a relation between the ω- bounded semicharacters on S and $\widetilde{\omega}$- bounded semicharacters on $M_\omega(S)$ may be compared with the Gelfand space of a commutative Banach algebra $\mathcal{A}$ and its multiplier algebra $M(\mathcal{A})$.

Theorem 2.15 ((2), Theorem 3.2)**.** *If $\alpha \in \Phi_{\omega s}(S)$, then there exists an unique $\widetilde{\alpha} \in \Phi_{\widetilde{\omega} s} M_\omega(S)$ such that $\widetilde{\alpha}(\gamma_s) = \alpha(s)$ for all $s \in S$. If $\beta \in \Phi_{\widetilde{\omega} s}(M_\omega(S))$, then either $\beta(\gamma_s) = 0$ for all $s \in S$ or there is $\alpha \in \Phi_{\omega s}(S)$ such that $\beta = \widetilde{\alpha}$. So, $\Phi_{\widetilde{\omega} s}(M_\omega(S)) = \Phi_{\omega s}(S) \cup h_{\omega s}(S)$, where $h_{\omega s}(S) = \{\beta \in \Phi_{\widetilde{\omega} s}(M_\omega(S)) : \beta(\gamma_s) = 0 \ (s \in S)\}$.*

As a consequence we have

Corollary 2.16 ((2), Corollary 3.4)**.** *Let (S,ω) be a weighted abelian semigroup. Then the following statements hold.*

(1) *The Gelfand space $\Delta(\ell^1(S,\omega))$ of $\ell^1(S,\omega)$ is homeomorphic to $\Phi_{\omega s}(S)$ with the point open topology.*
(2) *The Gelfand space $\Delta(\ell^1(M_\omega(S),\widetilde{\omega}))$ of $\ell^1(M_\omega(S),\widetilde{\omega})$ is homeomorphic to $\Phi_{\omega s}(S) \cup h_{\omega s}(S)$.*
(3) *Let ω be weakly regular. Then $\Delta(\ell^1(M_\omega(S)/S,\widetilde{\omega}_q))$ is homeomorphic to $h_{\omega s}(S)$.*

Semisimplicity of a Beurling algebra is an important problem. For a locally compact group G, $L^1(G,\omega)$ is semisimple if G is abelian [4]; for non-abelian G, it is not known whether $L^1(G,\omega)$ is semisimple. For an abelian semigroup S, $\ell^1(S,\omega)$ is semisimple if and only if S is separating and ω is semisimple [9]. This gives the following.

Theorem 2.17 ((2), Theorem 3.5)**.** *The Banach algebra $\ell^1(S,\omega)$ is semisimple if and only if $\ell^1(M_\omega(S),\widetilde{\omega})$ is semisimple. The quotient $\ell^1(M_\omega(S),\widetilde{\omega})/\ell^1(S,\omega)$ may fail to be semisimple.*

A *uniform norm* on a Banach algebra $(\mathcal{A}, \|\cdot\|)$ is a norm $|\cdot|$ satisfying $|x^2| = |x|^2$ $(x \in \mathcal{A})$. A Banach algebra has *UUNP (unique uniform norm property)* [6] if it has exactly one uniform norm. The UUNP is closely related with regularity [5, 13] and have applications to abelian harmonic analysis [14, 3, 8]. For an abelian G, the algebra $L^1(G)$ is regular; and for a weighted group G, $L^1(G,\omega)$ is regular iff $L^1(G,\omega)$ has UUNP iff ω is a Beurling-Domar weight [5]. It is interesting to search for a weighted abelian semigroup (S,ω) such that $\ell^1(S,\omega)$ has UUNP but is not regular. In the case of (S,ω), we have the following result and it may be compared with results on a commutative Banach algebra $\mathcal{A}$ and its multiplier algebra $M(\mathcal{A})$.

Theorem 2.18 ((2), Theorem 3.6)**.** *Let (S,ω) be a weighted abelian semigroup. Then the following statements hold.*

(1) *If* $\ell^1(M_\omega(S), \widetilde{\omega})$ *has UUNP, then* $\ell^1(S,\omega)$ *has UUNP.*

(2) *If* $\ell^1(M_\omega(S), \widetilde{\omega})$ *is regular, then* $\ell^1(S,\omega)$ *is regular.*

(3) *Let* S *be an inverse semigroup. Let* ω *be a Beurling-Domar weight on* S*. Then* $\ell^1(S,\omega)$ *is regular.*

A Banach $*$- algebra $(\mathcal{B}, \|\cdot\|)$ has *unique* C^**- norm property (UC*NP)* [2] if $\mathcal{B}$ admits exactly one C^*- norm. A commutative Banach $*$- algebra $\mathcal{B}$ is $*$- regular [2] if F is a closed subset of the hermitian Gelfand space $\widetilde{\Delta}(\mathcal{B})$ and $\psi \in \widetilde{\Delta}(\mathcal{B}) \setminus F$, there exists $x \in \mathcal{B}$ such that $\widehat{x}(\psi) \neq 0$ and $\widehat{x}(F) = \{0\}$. In fact, UC*NP and $*$- regularity (appropriately defined) acquires much greater significance in non-commutative Banach $*$- algebras [2]. By [11], for a weighted compactly generated (not necessarily abelian) group (G,ω), $L^1(G,\omega)$ is symmetric if and only if ω is a GRS- weight. By [2], a commutative Banach $*$- algebra is regular if and only if it is $*$- regular and symmetric. We have the following result for an involutive semigroup S having a symmetric weight ω, i.e., $\omega(s) = \omega(s^*)$ $(s \in S)$.

Theorem 2.19 ((2), Theorem 3.7)**.** *Let* S *be involutive, and let* ω *be symmetric. Then the following statements hold.*

(1) *If* $\ell^1(M_\omega(S), \widetilde{\omega})$ *has UC*NP, then* $\ell^1(S,\omega)$ *has UC*NP.*

(2) *If* $\ell^1(M_\omega(S), \widetilde{\omega})$ *is* $*$*- regular, then* $\ell^1(S,\omega)$ *is* $*$*- regular.*

As is said earlier that the interrelation between the Banach algebra structure of $\ell^1(S,\omega)$ and the structure of (S,ω) is a fascinating aspect of harmonic analysis. Now, we shall describe the multiplier algebra of $\ell^1(S,\omega)$.

Let $(\mathcal{A}, \|\cdot\|)$ be a commutative Banach algebra. A bounded linear map $T : \mathcal{A} \to \mathcal{A}$ is a multiplier on $\mathcal{A}$ if $T(ab) = aTb = (Ta)b$ for all $a, b \in \mathcal{A}$. Let $M(\mathcal{A})$ be the collection of all multipliers on $\mathcal{A}$. Then $M(\mathcal{A})$ is a commutative unital Banach algebra with composition as multiplication and the operator norm $\|T\| = \sup\{\|Ta\| : a \in \mathcal{A}, \|a\| \leq 1\}$ $(T \in M(\mathcal{A}))$.

These results of identifying the multiplier algebra of $\ell^1(S,\omega)$ are inspired by [16, 17], in which he considered the case of semigroups without weights. Let (S,ω) be a weighted abelian semigroup with zero element 0. The *annihilator* S^0_ω of S in $M_\omega(S)$ is a semigroup ideal of $M_\omega(S)$ given by

$$S^0_\omega = \{\alpha \in M_\omega(S) : \alpha\gamma_s = 0 (s \in S)\}.$$

Clearly, $0 \in S^0_\omega$. Analogously, the annihilator $\ell^1(S,\omega)^0$ of $\ell^1(S,\omega)$ in $\ell^1(M_\omega(S), \widetilde{\omega})$ is a closed algebra ideal of $\ell^1(M_\omega(S), \widetilde{\omega})$ given by

$$\ell^1(S,\omega)^0 = \{\mu \in \ell^1(M_\omega(S), \widetilde{\omega}) : \mu \star f = 0 \ (f \in \ell^1(S,\omega))\}.$$

When S is a semigroup with zero element 0, $M_\omega(S)$ is also a semigroup having zero element γ_0. Also, $\alpha(0) = 0$ for all $\alpha \in M_\omega(S)$. When S has a zero element, we define

$$\ell^1(S,\omega) = \{f : S \to \mathbb{C} : f(0) = 0, \|f\|_\omega = \sum_{s\in S} |f(s)|\omega(s) < \infty\}.$$

Clearly, $\ell^1(S,\omega)$ is a convolution Banach algebra. To get multiplier algebra of $\ell^1(S,\omega)$ we shall require the following results.

Lemma 2.20 ((6), Lemma 2.1). *Let S be an abelian faithful semigroup. Then the natural homomorphism $s \mapsto \gamma_s$ of S into $M_\omega(S)$ induces a homomorphism of $\ell^1(S,\omega)$ into $\ell^1(M_\omega(S),\widetilde{\omega})$ which is one to one if and only if $s \mapsto \gamma_s$ is one to one and onto if and only if $s \mapsto \gamma_s$ is onto.*

For $\mu \in \ell^1(M_\omega(S),\widetilde{\omega})$, let $T_\mu : \ell^1(S,\omega) \to \ell^1(S,\omega)$ be defined as

$$T_\mu(f) = \mu \star f \qquad (f \in \ell^1(S,\omega)).$$

Lemma 2.21 ((6), Lemma 2.2). *Let ω be a weight on an abelian semigroup S and let $\mu \in \ell^1(M_\omega(S),\widetilde{\omega})$. Then the map $T_\mu : \ell^1(S,\omega) \to \ell^1(S,\omega)$ defined by $T_\mu(f) = \mu \star f$ ($f \in \ell^1(S,\omega)$) is a multiplier on $\ell^1(S,\omega)$. The map $\mu \mapsto T_\mu$ of $\ell^1(M_\omega(S),\widetilde{\omega})$ into $M(\ell^1(S,\omega))$ is a norm-decreasing homomorphism.*

The following identifies multiplier algebra of $\ell^1(S,\omega)$ for abelian cancellative S.

Theorem 2.22 ((6), Theorem 1.1). *Let S be cancellative. Then $M(\ell^1(S,\omega))$ is homeomorphically isomorphic to $\ell^1(M_\omega(S),\widetilde{\omega})$.*

The following gives relation between these annihilators.

Theorem 2.23 ((6), Theorem 1.2). *Let S be an abelian semigroup with zero element. Let $\widetilde{\omega}$ (in particular, ω) be bounded away from 0. Then $\ell^1(S,\omega)^0 = \ell^1(S^0_\omega,\widetilde{\omega})$ and $\ell^1(M_\omega(S),\widetilde{\omega})/\ell^1(S^0_\omega,\widetilde{\omega})$ is isomorphic to the Beurling algebra $\ell^1(M_\omega(S)/S^\omega_0,\widetilde{\omega}_q)$.*

Now, if S is not necessarily cancellative, then under some mild conditions on S, ω and $\ell^1(S,\omega)$ we may identify the multiplier algebra of $\ell^1(S,\omega)$.

Theorem 2.24 ((6), Theorem 1.3). *Let S be separating and ω be semisimple, and let ω be bounded away from 0. Then the following hold.*

(1) *The map $f \mapsto f + \ell^1(S,\omega)^0$ from $\ell^1(S,\omega)$ into $\ell^1(M_\omega(S),\widetilde{\omega})/\ell^1(S,\omega)^0$ is one to one and $\ell^1(M_\omega(S),\widetilde{\omega})/\ell^1(S,\omega)^0$ is semisimple.*

(2) *If $\ell^1(S,\omega)$ has a bounded approximate identity, then the map $\mu + \ell^1(S,\omega)^0 \mapsto T_\mu$ is a homeomorphic isomorphism from $\ell^1(M_\omega(S),\widetilde{\omega})/\ell^1(S,\omega)^0$ onto $M(\ell^1(S,\omega))$.*

My Research Papers with Professor Bhatt

(1) *Beurling algebra analogues of theorems of Wiener-Lévy-Żelazko and Żelazko*, Studia Mathematica, 195(3)(2009) 219 - 225 (jointly with H. V. Dedania).

(2) *Multipliers of weighted semigroups and associated Beurling Banach algebras*, Proc. Indian Acad. Sci. (Math. Sci.), **121**(4)(2011) 417 - 433 (jointly with H. V. Dedania).

(3) *Absolutely differentiable functions and functions of bounded differential variation*, The Math. Student, **79**(1- 4)(2011) 237–251 (jointly with Krunal Kachhia).

(4) *Arens regularity and Amenability of Lau product of Banach algebras defined by a Banach algebra morphism*, Bull. Aust. Math. Soc., **87**(2)(2013) 195–206.

(5) *The $*$- semisimplicity of the ℓ^1-algebra on an abelian $*$-semigroup*, Bull. Aust. Math. Soc., **88**(3)(2013) 492–498 (jointly with H. V. Dedania).

(6) *The multiplier algebra of a Beurling algebra*, Bull. Aust. Math. Soc., **90**(1)(2014), 113–120 (jointly with H. V. Dedania).

(7) *On an intrinsic characterization of self-adjoint C^*-Segal algebras*, Colloq. Mathematicum, **145**(2) 245–250.

(8) *Segal extensions and Segal algebras in uniform Banach algebras*, to appear in Indian Journal of Pure and Applied Mathematics.

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