Mathematics Today Volume 36 Issue 1 (September 2020), 18-35
ISSN 0976-3228, e-ISSN 2455-9601

UNIQUENESS PROPERTIES IN SEMISIMPLE, COMMUTATIVE, BANACH ALGEBRAS

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1. Tribute

Twenty eight years ago, one day in the summer vacation of May 1992, I came to Vallabh Vidyanagar to seek Ph.D. admission in the Department of Mathematics, Sardar Patel University, Vallabh Vidyanagar. It was on that day that I met Professor Subhash J. Bhatt for the first time. In July 1992, I was given admission under his guidance. I was very fortunate that, though Professor Bhatt was my advisor, our student-teacher relation turned into one of close friends over the period of time. I have written nineteen joint papers with him. I feel privileged and proud to record here that he has highest number of joint papers with me. We had nearly perfect tuning and bonding. Its secret might lie in our similar rural background in childhood. My assessment is that he contributed more to the department than any other past faculty as long as the department's projects and infrastructure are concerned. His knowledge was not limited to functional analysis only. He did research in applied mathematics, financial mathematics, mathematical physics, mathematical biology, and in some other areas too. He was interested in even Sanskrit language and used to attend seminar or conference at the Department of Sanskrit.

2. Our Main Research Areas

My collaborative research with him is mainly in the general theory of commutative Banach algebras, the Beurling algebra $L^1(G,\omega)$ on a locally compact group G, the weighted discrete semigroup algebra $\ell^1(S,\omega)$, and the Fourier series.

3. Our First Three Research Papers

After joining Ph.D., my first paper jointly with him was published in 1992. Its title was "On a problem of H. G. Dales in Banach algebra". I have mixed feeling to note here that, at later stage, I had to discontinue my Ph.D. registration under his guidance and registered under Professor H. G. Dales at the University of Leeds, UK, in October 1994 to avail the Commonwealth Scholarship.

2010 *Mathematics Subject Classification.* 46J05, 22B10.
Key words and phrases. Commutative Banach Algebras, Semisimplicity, Uniform Norm, Spectral Norm, C^*-norm, Spectral Extension Property, Regularity, Weakly Regular, Locally Compact Abelian Group, Measurable Weight, Beurling Algebras.

Anyway, the paper was about proving the existence of a complete algebra norm on an algebra by some algebraic conditions. Note that if X is a vector space, then there always exists a linear norm on X but there can not be complete linear norm if X has a countably infinite (Hamel) basis. On the other hand, if A is an algebra, then there may not be any algebra norm on A. If A admits an algebra norm (resp., complete algebra norm), then A is called a *normable (resp., Banachable) algebra*. Following two results are main among others in this paper.

Proposition 3.1 (1; Page-32). *Let A be a semisimple algebra over the complex field $\mathbb{C}$.*

(1) *A is Banachable iff it is a homomorphic image of a Banach algebra.*
(2) *Let A be a division algebra which is a homomorphic image of a normed algebra. Then A is isomorphic to $\mathbb{C}$.*

By an *algebraic set*, we mean, a subset that is absolutely convex, absorbing, idempotent set. A subset E of a vector space X is *radially bounded*, if for any given $x \neq 0$ in X, the subset $K(x) = \{r \in \mathbb{R} : rx \in E\}$ of $\mathbb{R}$ is either empty or bounded [BD, Page 3]. A radially bounded, algebraic set is *completant* [Wa, Page 23] if its Minkowski functional is a complete norm.

Proposition 3.2 (1; Page-32). *Let (A, τ) be a topological algebra. Then*

(1) *A is Banachable that admits a complete algebra norm including a topology finer than τ iff there exists a bounded completant algebraic set.*
(2) *A is Banachable that admits a topology weaker than τ iff there exists a completant, radially bounded, balanced, convex neighbourhood of zero which is also an idempotent.*

Our second paper is on the characterization of a linear norm under different conditions being equivalent to the supremum norm $\|\cdot\|_\infty$ on $C(X)$. My co-guide Prof. M. H. Vasavada was also co-author in this paper. It was this paper which set my research direction. The following are our main results in this paper.

Theorem 3.3 (2; Theorems 1). *Let $C(X)$ be the algebra of all complex-valued, continuous functions on a compact, Hausdorff space X. Let $\|\cdot\|$ be a linear norm on $C(X)$. Then $\|\cdot\|$ and $\|\cdot\|_\infty$ are equivalent on $C(X)$ if any one of the following conditions is held.*

(1) *There exist constants $c, k > 0$ such that $c\|f\|^2 \leq \|f^2\| \leq k\|f\|^2$ $(f \in C(X))$.*
(2) *There exist constants $c, k > 0$ such that $c\|f\|^2 \leq \|f^*f\| \leq k\|f\|^2$ $(f \in C(X))$.*

Theorem 3.4 (2; Theorems 2). *Let $\|\cdot\|$ be an algebra norm on $C(X)$. $[f] \in C(X)$ such that $[f] \geq 0$ on X and $[f]^2 = f^*f$.*

(1) *If the involution $*$ is $\|\cdot\|$-continuous, and if there exists $k > 0$ such that $k\|f\| \leq \|[f]\|$ $(f \in C(X))$ (in particular, for some $c, k > 0$, $c\|f\| \leq \|[f]\| \leq k\|f\|$ $(f \in C(X))$), then $\|\cdot\|$ is equivalent to $\|\cdot\|_\infty$.*

(2) *If* $\|1_X\| = 1$, $\|f\| = \|[f]\|$ $(f \in C(X))$, *then* $\| \cdot \| = \| \cdot \|_\infty$.

The non-commutative version of the above results is as follow.

Theorem 3.5 (2; Theorems 3)**.** *Let* $(A, \| \cdot \|)$ *be a unital* C^**-algebra.*

(1) *Let* $| \cdot |$ *be any norm on* A *such that* $(A, | \cdot |)$ *is a normed linear space.*
  (a) *If* $|x|^2 = |x^* x|$ $(x \in A)$, *then* $| \cdot | = \| \cdot \|$ *on* A.
  (b) *If the involution* $*$ *is* $| \cdot |$*-continuous, and if there exists* $k > 0$ *such that* $k|x|^2 \leq \|x^* x\|$ $(x \in A)$, *then* $| \cdot |$ *is equivalent to* $\| \cdot \|$.

(2) *Let* $| \cdot |$ *be an algebra norm on* A *and let the involution* $*$ *be* $| \cdot |$*-continuous.*
  (a) *If* $|1| = 1$, $|x| = |[x]|$ $(x \in A)$, *then* $| \cdot | = \| \cdot \|$ *on* A.
  (b) *If there exists* $k > 0$ *such that* $k|x| \leq |[x]|$ $(x \in A)$, *then* $| \cdot | \cong \| \cdot \|$.

The concept in our third paper is well-known; namely, the topological zero divisor. The article is simple and small but the results are surprising. Motivated by this paper, Schulz, Brits, and Hasse have published a paper [SBH]. A question is that which Banach algebras have every element topological zero divisor? Note that such Banach algebras should be necessarily non-unital. Here is our main result.

Theorem 3.6 (3; Page-735)**.** *Every element of a complex Banach algebra* $(A, \|\cdot\|)$ *is a topological divisor of zero (TDZ) if at least one of the following holds.*

(1) *A is infinite dimensional and it admits an orthogonal basis.*

(2) *A is a non-unital, uniform Banach algebra in which the Silov boundary* ∂A *coincides with the Gel'fand space* $\Delta(A)$*. In particular, A is a non-unital, regular, uniform Banach algebra.*

(3) *A is a non-unital, hermitian, Banach* $*$*-algebra with a continuous algebra involution. In particular, A is a non-unital* C^**-algebra.*

4. Uniqueness of Uniform Norms

As I said earlier, the second paper set our common research area. It is about different types of (algebra) norms on Banach algebras and their uniqueness. A detailed account is as follow.

Throughout, A is an associative algebra over the complex field $\mathbb{C}$. An algebra (semi-)norm on A will simply be called a (semi-)norm. When A is a normed (or Banach) algebra, the notation $\| \cdot \|$ is reserved for that normed (or Banach) algebra norm on A. Either our papers or H. G. Dales' book [Da] will be the main references for notation, terminology, and definition.

A norm $| \cdot |$ (not necessarily complete) on an algebra A is: (i) a *uniform norm* if it satisfies the square property $|x^2| = |x|^2$ $(x \in A)$; (ii) a *spectral norm* if $r(x) \leq |x|$ $(x \in A)$; (iii) a *semisimple norm* if the completion of $(A, | \cdot |)$ is a semisimple Banach algebra. If A admits at least one uniform norm, then A is necessarily semisimple and commutative. The converse holds if A is a Banach algebra. Moreover, any two equivalent uniform semi-norms on A are identical. The spectral radius $r(\cdot)$ is the largest uniform seminorm on a commutative Banach algebra, which is a norm iff A is semisimple. A Banach algebra A has *unique uniform norm property (UUNP)* if A admits exactly one uniform norm. A Banach algebra A has *spectral extension property (SEP)* if every norm on A is spectral. Let $F \subset \Delta(A)$. Then $|x|_F = \sup\{|\varphi(x)| : \varphi \in F\}$ is always a uniform semi-norm on A. The set F is a *set of uniqueness for A* if $|\cdot|_F$ is a norm on A. Let $r_p(x) = \inf\{|x| : |\cdot| \text{ is a norm on } A\}$ $(x \in A)$ and $r_s(x) = \inf\{|x| : |\cdot| \text{ is a semisimple norm on } A\}$ $(x \in A)$, which are called respectively *permanent radius* and *semisimple permanent radius* on A. The following theorem gives characterization of UUNP under different conditions and they are used frequently to prove results on UUNP.

Theorem 4.1. *(6; Theorem 2.3) Let A be a semisimple, commutative Banach algebra. Then the following are equivalent.*

(1) *A has UUNP;*
(2) *The Silov boundary ∂A is the smallest closed set of uniqueness for A.*
(3) *If $F \subset \Delta(A)$ is closed and it does not contain ∂A, then there exists $a \in A$ such that $r(a) > 0$ and $\widehat{a} = 0$ on F.*
(4) *If $F \subset \Delta(A)$ is closed and it does not contain ∂A, then there exists $a \in A$ such that $r_s(a) > 0$ and $\widehat{a} = 0$ on F.*
(5) *Every complex homomorphism φ in ∂A belongs to $\Delta(B)$, where B is any semisimple, commutative Banach algebra containing A as a subalgebra.*
(6) *A has semisimple SEP (i.e. every semisimple norm is spectral).*
(7) *$r(x) = r_s(x)$ $(x \in A)$.*
(8) *The topological boundary of the spectrum $\sigma_A(a)$ is contained in $\sigma_B(a) \cup \{0\}$ for every $a \in A$, where B is any semisimple, commutative Banach algebra containing A as a subalgebra.*

Theorem 4.2. *(6; Proposition 2.4) Let A be a semisimple, commutative Banach algebra. Then the following are equivalent.*

(1) *A has SEP;*
(2) *A has UUNP and $r_s(x) = r_p(x)$ $(x \in A)$;*
(3) *A has UUNP and every norm on A dominates a semisimple norm on A.*
(4) *A has UUNP and A has P-property (i.e. every non-zero closed ideal I of A contains an element a such that $r_p(a) > 0$.*

Proposition 4.3 (6; Proposition 2.6)**.** *Let B be a dense ideal of $(A, \|\cdot\|)$ such that B is a Banach algebra with some norm $|\cdot|$. If A has UUNP, then B has UUNP.*

(1) *A has SEP;*

(2) *A has UUNP and* $r_s(x) = r_p(x)$ $(x \in A)$*;*

(3) *A has UUNP and every norm on A dominates a semisimple norm on A.*

(4) *A has UUNP and A has P-property (i.e. every non-zero closed ideal I of A contains an element a such that* $r_p(a) > 0$.

Open Problem: It is clear from above two theorems that SEP implies UUNP. M. J. Meyer claimed in [Me, Theorem 1] that A has SEP iff the Silov boundary ∂A is the smallest closed set of uniqueness for A. As a corollary, it follows that UUNP and SEP are equivalent. However, there is a gap in the proof. So it is still open question that: Does UUNP imply SEP?

In trying to solve the above problem, we proved several results on UUNP. Moreover, we established links between UUNP and other Banach algebra properties. So our second paper on UUNP encouraged us to study it in depth. I am pleased to report here that these results are included in Springer's graduate textbook written by Kaniuth [Ka].

For an ideal I in A and a subset F of $\Delta(A)$, let $h(I) \subset \Delta(A)$ and $k(F) \subset A$ denote the hull of I and the kernel of F respectively. An ideal I of A is a *spectral synthesis ideal (or a semisimple ideal)* if $I = k(h(I))$; equivalently, the quotient algebra A/I is semisimple.

Proposition 4.4 (7, Proposition 2.1)**.** *Let A be a semisimple, commutative Banach algebra and* $U(A)$ *be the completion of the normed algebra* $(A, r(\cdot))$*. Then the following are equivalent.*

(1) *A has UUNP;*

(2) $U(A)$ *has UUNP and every closed subset F of* $\Delta(U(A))$ *which is a set of uniqueness for A is also a set of uniqueness for* $U(A)$.

(3) $U(A)$ *has UUNP and* $I \cap A \neq \phi$ *for any non-zero spectral synthesis ideal I of* $U(A)$.

Proposition 4.5 (7, Proposition 2.3)**.** *Let A have UUNP and I be a closed ideal of A. If I is a spectral synthesis ideal of A, or if* $\overline{I}$ *is a spectral synthesis ideal of* $U(A)$*, then I has UUNP. The quotient algebra* A/I *need not have UUNP.*

Lemma 4.6 (7, Lem 2.4)**.** *If some dense subalgebra B of A has UUNP, then A has UUNP.*

Lemma 4.7 (7, Lemma 2.5)**.** *Let I be a spectral synthesis ideal of* $U(A)$*. If* $A/(A \cap I)$ *has UUNP, then* $I = k(h(A \cap I))$.

A semisimple, commutative Banach algebra A is: (i) *regular* if, for every closed set $F \subset \Delta(A)$ and $\psi \in \Delta(A) \setminus F$, there exists (necessarily non-zero) $a \in A$ such that $\varphi(a) = 0$ $(\varphi \in F)$ and $\psi(a) \neq 0$; (ii) *weakly regular* if, for every proper closed subset F of $\Delta(A)$, there exists a non-zero element $a \in A$ such that $\varphi(a) = 0$ $(\varphi \in F)$.

Theorem 4.8 (7, Theorem 2.6)**.** *Let A be a semisimple, commutative Banach algebra. Then the following are equivalent.*

(1) *A is regular;*

(2) *$U(A)$ is regular and the quotient algebra A/I has UUNP for every spectral synthesis ideal I of A.*

(3) *$U(A)$ is regular and $I = k(h(A \cap I))$ for every spectral synthesis ideal I of $U(A)$.*

A semisimple, commutative Banach algebra A is an *N-algebra* [Ri, Page 92] if every closed ideal of A is a spectral synthesis ideal.

Proposition 4.9 (7, Proposition 2.7)**.** *Consider the following statements.*

(1) *A is regular;*

(2) *$U(A)$ is regular and, for every non-zero spectral synthesis ideal I of $U(A)$, $A \cap I$ is dense in A.*

Then (2) implies (1). Further, if $U(A)$ is an N-algebra, then (1) implies (2).

By (6, Page 581), if $\dim(A) > 1$ and A has UUNP, then A cannot be an integral domain. The following shows that its converse is not true.

Proposition 4.10 (7, Proposition 2.8)**.** *There exists a semisimple, commutative Banach algebra A having the following properties.*

(1) *$dim(A) > 1$;*

(2) *A does not have UUNP;*

(3) *A is not an integral domain.*

Proposition 4.11 (7, Proposition 2.9)**.** *Let $(B, \|\cdot\|_B)$ be a unital, semisimple, commutative Banach algebra, $\Omega = \Delta(B) \times [0,1]$, $A = \{f \in C(\Omega) : f(\varphi, 0) = \widehat{b}(\varphi)(\varphi \in \Delta(B))$ for some $b \in B\}$ with the norm $\|f\| = \max\{\|f\|_\Omega, \|f(\cdot, 0)\|_B\}$, where $\|\cdot\|_\Omega$ denotes the supnorm on Ω. Then the following hold.*

(1) *$(A, \|\cdot\|)$ is a unital, semisimple, commutative Banach algebra with pointwise operations.*

(2) *$\Delta(A) \cong \Omega$ and $\partial A = \Delta(A)$.*

(3) *A has SEP and hence it has UUNP.*

(4) *A is always weakly regular.*

(5) *A is an uB-algebra iff B is an uB-algebra iff $\|\cdot\| = \|\cdot\|_\Omega$ on A.*

(6) *A is regular iff B is regular.*

(7) *A is hermitian with complex conjugation iff B is hermitian with some involution.*

A *C^*-norm* on a $*$-algebra A is a (not necessarily complete) norm $|\cdot|$ on A such that $|x^*x| = |x|^2$ $(x \in A)$. The A has *unique C^*-norm property (UC*NP)* if A admits exactly one C^*-norm. Then the following proposition gives the relation between UUNP and UC^*NP. Note that every C^*-norm on a commutative $*$-algebra A is a uniform norm. Hence it follows from the following proposition that every $*$-semisimple, non-hermitian, commutative Banach $*$-algebra has more than one uniform norms.

Proposition 4.12 (7, Proposition 2.10). *Let A be a commutative Banach $*$-algebra. Then*
(I) Any two of the following imply the third:

(1) *A has UUNP;*

(2) *A has UC*NP.*

(3) *A is hermitian.*

(II) Let A be $*$-semisimple. Then the following holds.

(1) *If A has UUNP, then A has UC*NP and A is hermitian.*

(2) *A is weakly regular iff A is hermitian and A has UUNP.*

Proposition 4.13 (7, Proposition 2.13). *Let $\Delta(A)$ be homeomorphic to either a subset of $\mathbb{R}$ or a subset of the unit circle in the complex plane. Then A is regular iff A is weakly regular.*

Let A and B be $*$-semisimple Banach $*$-algebras. Let $C^*(A)$ and $C^*(B)$ be C^*-enveloping algebras. By [HKV, Theorems 3.3 and 3.4], the algebraic tensor product $A \otimes B$ has UC^*NP iff each of A, B, and $C^*(A) \otimes C^*(B)$ has UC^*NP. However, we get different result for UUNP.

Theorem 4.14 (7, Theorem 3.1). *Let A and B be semisimple, commutative Banach algebras. Then $A \otimes B$ has UUNP iff both A and B have UUNP.*

Corollary 4.15 (7, Corollary 3.2). *Let α be a submultiplicative norm on $A \otimes B$ and let $A \otimes_\alpha B$ be the completion of $(A \otimes B, \alpha)$ such that $A \otimes_\alpha B$ is semisimple. If both A and B have UUNP, then $A \otimes_\alpha B$ has UUNP.*

A *multiplier* on an algebra A is a linear map $T : A \longrightarrow A$ such that $T(xy) = (Tx)y = xT(y)$ $(x, y \in A)$. Let $M(A)$ be the set of all multipliers on A. If A is a semisimple, commutative Banach algebra, then $M(A)$ is also a semisimple, commutative Banach algebra with respect to linear operations, composition, and the operator norm. The *derived algebra* A_0 of A is defined as $A_0 = \{x \in A : f\widehat{x} \in \widehat{A} \text{ for all } f \in C_0(\Delta(A))\}$. If A is self-adjoint (i.e., having a hermitian involution), then $A_0 = \{x \in A : \sup\{\|xy\| : y \in A, r(y) \leq 1\} < \infty\}$. We shall refer to [La] for more details on multipliers. We proved the following results on multipliers.

Theorem 4.16 (7, Theorem 6.1). *Let A be a semisimple, commutative Banach algebra.*

(1) *If $M(A)$ has UUNP, then A has UUNP.*

(2) *Let A be a uB-algebra. Then A has UUNP iff $M(A)$ has UUNP.*

(3) *If A is hermitian, then A_0 has UUNP.*

(4) *If A is weakly regular or if A_0^2 is dense in A_0, them $M(A_0)$ has UUNP.*

(5) *Let A be an H^*-algebra and let $K(A)$ denote the algebra of compact multipliers on A. Then each of A, $M(A)$, and $K(A)$ has UUNP.*

Proposition 4.17 (7, Proposition 6.2). *Let A be a semisimple, commutative Banach algebra. Then*

(1) $|T|_\infty = \sup\{|\widehat{T}(\varphi)| : \varphi \in \Delta(A)\}$ $(T \in M(A))$ *is a uniform norm on* $M(A)$.

(2) *The spectral radius* $r_{M(A)}(\cdot)$ *is the largest uniform norm on* $M(A)$.

(3) *Let A be weakly regular. Then* $|T|_\infty \leq |T| \leq r_{M(A)}(T)$ $(T \in M(A))$ *for every uniform norm* $|\cdot|$ *on* $M(A)$.

Corollary 4.18 (7, Corollary 6.3). *Let A be weakly regular. If $\Delta(A)$ is a set of uniqueness for $M(A)$, then the $|\cdot|_\infty$ is the smallest uniform norm on $M(A)$.*

Let $M_{00}(A) = \{T \in M(A) : \widehat{T} = 0 \text{ on } h(A)\} = k(h(A))$ in $M(A)$ and $M_0(A) = \{T \in M(A) : \widehat{T}_{|\Delta(A)} \in C_0(\Delta(A))\}$. Then both are closed ideals in $M(A)$ and $A \subset M_{00}(A) \subset M_0(A) \subset M(A)$. These ideals are defined in [LN].

Corollary 4.19 (7, Corollary 6.4). *Let A be weakly regular.Then $M_{00}(A)$ has UUNP, but $M_0(A)$ need not have UUNP.*

We know that there are several semisimple, commutative Banach algebras without identity. Fortunately, we can adjoin an identity in any Banach algebra as follow. Let $A_e = A \times \mathbb{C}$. For $(x, \alpha), (y, \beta) \in A_e$, define $(x, \alpha)(y, \beta) = (xy + \alpha x + \beta y, \alpha\beta)$. Then A_e is a unital Banach algebra with linear operations, abover product, and the norm $\|(x, \alpha)\| = \|x\| + |\alpha|$. So it is natural to ask which Banach algebra properties are preserved while adjoining an identity? Following are our results in this direction.

Theorem 4.20. *Let A be a non-unital, semisimple, commutative Banach algebra and let A_e be the unitization of A. Then the following holds.*

(1) *(4, Theorem 3.1) A has UUNP iff A_e has UUNP.*

(2) *(4, Corollary 3.2) A has SEP iff A_e has SEP.*

(3) *(4, Corollary 3.3) A is weakly regular iff A_e weakly regular.*

There are some important algebra seminorms on Banach algebras which are not norms. For example, the spectral radius on commutative Banach algebras and the largest C^*-seminorm on a Banach $*$-algebra. By the same time, in proving equivalence or uniqueness of norms, one may need weaker conditions than the "square property" and the "C^*-property". Following are the results in more general set up.

Lemma 4.21 (5, Lamma on Page 552). *Let $\|\cdot\|$ be a seminorm on an algebra A. Then the following hold.*

(1) *Let* $c > 0$ *such that* $\|x^2\| \leq c\|x\|^2$ $(x \in A)$. *Then* $\|x^n\| \leq 3^{n-2}c^{n-1}\|x\|^n$ $(x \in A;\ n \geq 2)$. *If* A *is commutative, then* $\|xy\| \leq 3c\|x\|\|y\|$ $(x, y \in A)$.

(2) *If* $\|\cdot\|$ *satisfies either the square property or the power property, then* $\|x^n\| = \|x\|^n$ $(x \in A;\ n \in \mathbb{N})$.

(3) *Let* $\|\cdot\|$ *satisfy the square inequality. Then* $\|\cdot\|$ *is weakly submultiplicative,* $\|\cdot\|$ *is equivalent to some algebra seminorm with square property, and* A/N *is a commutative, semisimple algebra, where* $N = \{x \in A : \|x\| = 0\}$.

(4) *Let* A *be a* $*$*-algebra. Let* $\|\cdot\|$ *satisfy the* C^**-inequality. Then* $\|\cdot\|$ *is weakly submultiplicative, the involution* $*$ *is* $\|\cdot\|$*-continuous, and* $\|\cdot\|$ *is equivalent to some algebra seminorm with* C^**-property.*

Theorem 4.22 (5, Theorem on Page 551)**.** *Let* $\|\cdot\|$ *be a spectral linear seminorm on a Banach algebra* A *and let* $N = \{x \in A : \|x\| = 0\}$. *Then the following hold.*

(1) *If* $\|\cdot\|$ *satisfies the square inequality, then* $\|\cdot\|$ *is equivalent to the spectral radius* $r_A(\cdot)$, *the* N *is equal to the (Jacobson) radical of* A, *and* A/N *is commutative. If* $\|\cdot\|$ *satisfies the square property, then* $\|\cdot\| = r_A(\cdot)$.

(2) *Let* A *be a* $*$*-algebra. If* $\|\cdot\|$ *satisfies the* C^**-inequality, then* $\|\cdot\|$ *is equivalent to the Ptak's spectral function* $s_A(x) = r_A(x^*x)^{1/2}$ $(x \in A)$, *the* N *is equal to the star radical of* A, *and* A *is hermitian. If* $\|\cdot\|$ *satisfies the* C^**-property, then* $\|\cdot\| = s_A(\cdot)$.

5. The Beurling Algebra $L^1(G, \omega)$

Our second area of research was the Beurling algebra $L^1(G, \omega)$ on a locally compact, Hausdorff, Abelian group G and the weighted discrete semigroup algebra $\ell^1(S, \omega)$. In this section, we discuss the results on $L^1(G, \omega)$.

Throughout let G be a locally compact abelian (LCA) group with the Haar measure λ and let $\widehat{G}$ denote the dual group of G, i.e., the set of all continuous group homomorphiosms $\theta : G \longrightarrow (\mathbb{T}, \times)$, where $\mathbb{T}$ is the unit circle in the complex plane; the elements of $\widehat{G}$ are called *characters* on G . Then it is well-known that $\widehat{G}$ is an LCA group in compact-open topology. I am happy to mention here that we have generalized this concept, which is called the generalized character on G. A *generalized character* on a topological group G is a continuous function $\alpha : G \longrightarrow (\mathbb{C}^\bullet, \times)$, where $\mathbb{C}^\bullet = \mathbb{C} \setminus \{0\}$ such that $\alpha(s + t) = \alpha(s)\alpha(t)$ $(s, t \in G)$. Let $H(G)$ denote the set of all generalized characters on G equipped with the compact-open topology. For $\alpha, \beta \in H(G)$, define $(\alpha + \beta)(s) = \alpha(s)\beta(s)$ $(s \in G)$. Then $(H(G), +)$ is an abelian topological group [HR, 23.34(b)]. It is straightforward to verify that $H(\mathbb{Z}) \cong (\mathbb{C}^\bullet, \times)$ and $H(\mathbb{T}) \cong (\mathbb{Z}, +)$. Let $C_C(G)$ denote the set of all complex-valued continuous functions on G with compact support. Then $C_C(G)$ is a commutative algebra with respect to the usual convolution product. Let τ

denote the inductive limit topology on $C_C(G)$. Then, by [M, Lemma 2.1, p.114], $(C_C(G), \tau)$ is a commutative topological algebra. Following are our main results in this paper.

Lemma 5.1 (12, Lemma 3.1)**.** *Let G be second countable, and let $f \in C_C(G)$. Then the map $\Lambda_f : G \longrightarrow (C_C(G), \tau); s \longmapsto \tau_s f$ is continuous.*

Theorem 5.2 (12, Theorem 3.2)**.** *Let G be second countable. Let $T : H(G) \longrightarrow \Delta(C_C(G))$ be defined as $T(\alpha) = \varphi_\alpha$, where $\varphi_\alpha(f) = \int_G f(s)\alpha(s)d\lambda(s)$ $(f \in C_C(G))$. Then T is a bijective continuous map.*

Definition 5.3. *For $\alpha \in H(G)$, $\varepsilon > 0$, and $\{f_1, \ldots, f_n\} \subseteq C_C(G)$, define*

$$B(\alpha; \varepsilon; f_1, \ldots, f_n) = \{\beta \in H(G) : |\widehat{f_i}(\beta) - \widehat{f_i}(\alpha)| < \varepsilon \ (1 \leq i \leq n)\},$$

where $\widehat{f}(\beta) = \varphi_\beta(f) = \int_G f(s)\beta(s)d\lambda(s)$. Then the collection

$$\mathcal{B} = \{B(\alpha; \varepsilon; f_1, \ldots, f_n) : \alpha \in H(G),\ n \in \mathbb{N},\ \{f_1, \ldots, f_n\} \subseteq C_C(G)\}$$

forms a basis for some topology on $H(G)$. Let τ_g denote the topology on $H(G)$ generated by this basis. Then $\tau_g \subseteq \tau_{co}$ on $H(G)$. Let $\widetilde{H}(G)$ denote the $H(G)$ equipped with the topology τ_g. We say that $\widetilde{H}(G) = H(G)$ if $\tau_{co} = \tau_g$.

Remark 5.4. *Let $r > 1$. Define $\omega(s) = e^{r|s|}$ $(s \in \mathbb{R})$. Then ω is a weight on $\mathbb{R}$ such that $\Delta(L^1(\mathbb{R}, \omega)) \cong \Pi_{-r,r} := \{x + iy \in \mathbb{C} : -r \leq x \leq r\}$ due to* [Da, Theorem 4.7.33, p.533].

Theorem 5.5. *Let G be an LCA group. Then*

1. *(12; Theorem 2.2) If G is compactly generated, then $H(G)$ is an LCA group.*
2. *(12; Theorem 2.2) Let G be compactly generated. Then $H(G) = \widehat{G}$ iff G is compact.*
3. *(12; Corollary 3.8) If G is second countable and compactly generated, then $H(G)$ is homeomorphic to $\Delta(C_C(G))$, and hence $\Delta(C_C(G))$ is locally compact.*
4. *(12; Corollary 3.7) Let G be compactly generated. Then $\widetilde{H}(G) = H(G)$.*
5. *(12; Theorem 3.9) If G is discrete, then $H(G) \cong \Delta(C_C(G))$.*

Let ω be a weight on G, i.e., a strictly positive, Borel measurable function ω on G such that $\omega(s + t) \leq \omega(s)\omega(t)$ $(s, t \in G)$. Then the *Beurling algebra* $L^1(G, \omega)$, consists of all complex-valued measurable functions f on G such that $f\omega \in L^1(G)$, is a commutative Banach algebra with respect to the convolution product and the weighted L^1-norm $\| f \|_\omega := \int_G |f(s)|\omega(s)d\lambda(s)$. An *$\omega$-bounded generalized character* on G is a generalized character α on G such that $|\alpha(s)| \leq \omega(s)$ $(s \in G)$. Let $H(G, \omega)$ denote the set of all ω-bounded generalized characters on G equipped with the compact-open topology. For $\alpha \in H(G, \omega)$, define

$\varphi_\alpha(f) = \widehat{f}(\alpha) = \int_G f(s)\alpha(s)d\lambda(s)$ $(f \in L^1(G,\omega))$. Then the map $T : H(G,\omega) \longrightarrow \Delta(L^1(G,\omega))$ defined as $T(\alpha) = \varphi_\alpha$ is a homeomorphism. It is easy to see that $\widehat{G} \subseteq H(G,\omega)$ iff $\omega \geq 1$ on G. In general, $H(G,\omega)$ need not be a group like $\widehat{G}$. However it is always true that $\theta\alpha \in H(G,\omega)$ $(\theta \in \widehat{G}$ and $\alpha \in H(G,\omega))$. For $F \subseteq H(G,\omega)$, define $|f|_F = \sup\{|\widehat{f}(\alpha)| : \alpha \in F\}$ $(f \in L^1(G,\omega))$. Then $|\cdot|_F$ is a uniform seminorm on $L^1(G,\omega)$; the F is a *set of uniqueness* for $L^1(G,\omega)$ if $|\cdot|_F$ is a norm. For example, $\alpha\widehat{G}$ is a set of uniqueness for any $\alpha \in H(G,\omega)$. For $\alpha \in H(G,\omega)$, the uniform norm $|\cdot|_{\alpha\widehat{G}}$ on $L^1(G,\omega)$ will be denoted by $|\cdot|_\alpha$. Our main results on the Beurling algebra $L^1(G,\omega)$ are following.

Theorem 5.6. *Let G be an LCA group and ω be a weight on G. Then*

(1) *(9, Theorem 1) $L^1(G,\omega)$ is always semisimple.*
(2) *(7, Theorem 4.1) $L^1(G,\omega)$ has UUNP iff it is regular.*
(3) *(7, Proposition 4.4) The Gel'fand space $\Delta(L^1(G,\omega))$ is homeomorphic to $H(G,\omega)$.*
(4) *(7, Theorem 4.5(1)) ω be symmetric iff $L^1(G,\omega)$ is a Banach $*$-algebra.*
(5) *(7, Theorem 4.5(2)) Let $L^1(G,\omega)$ be a Banach $*$-algebra. Then $L^1(G,\omega)$ is hermitian iff $\Delta(L^1(G,\omega)) \cong \widehat{G}$.*
(6) *(11, Theorem 1) $L^1(G,\omega)$ has UUNP iff it has a minimum uniform norm.*
(7) *(11, Theorem 2) $L^1(G,\omega)$ admits either exactly one uniform norm or infinitely many uniform norms.*
(8) *(7, Proposition 4.6) Let H be an additive subgroup of the rational group $\mathbb{Q}$ containing 1. Let ω be a weight on H such that $\omega(s) \geq 1$ $(s \in H)$. Then $\ell^1(H,\omega)$ has UUNP iff $\ell^1(\mathbb{Z},\omega)$ has UUNP.*

Open Problem It is still an open question whether $L^1(G,\omega)$ is semisimple for non-abelian, locally compact group G?

Remark 5.7. *Note that the last statement of the above Theorem was proved for arbitrary semisimple, commutative, Banach algebra by Dabhi and myself* [DaDe].

There are three more Banach algebras associated with the Beurling algebra $L^1(G,\omega)$; namely, $M_{00}(G,\omega)$, $M_0(G,\omega)$, and $M(G,\omega)$. In order to study the uniqueness properties of these algebras, we need the concept of *Radon Measure* which is defined in [Da, Appendix-4, Page-838]. A *Radon measure* on G is a continuous linear functional on $C_C(G)$. Let $M_{loc}(G)$ denote the linear space of all Radon measures on G. Let $L_{loc}(G)$ denote the space of all locally integrable, measurable functions on G. Then clearly $L_{loc}(G) \subseteq M_{loc}(G)$. Let $\mu \in M_{loc}(G)$. Define $D_{\mathcal{L}\mu} = \{\alpha \in H(G) : \int_G |\alpha(s)|d|\mu|(s) < \infty\}$. When $D_{\mathcal{L}\mu} \neq \phi$, the *Laplace transform* $\mathcal{L}\mu$ of μ (also denoted by $\widehat{\mu}$), is defined as

$$(\mathcal{L}\mu)(\alpha) = \widehat{\mu}(\alpha) = \int_G \alpha(s)d\mu(s) \ \ (\alpha \in D_{\mathcal{L}\mu}).$$

This introduces weighted analogues of the classical transforms of harmonic analysis, thereby providing important tools for abelian weighted harmonic analysis. Let $M(G)$ be the convolution Banach algebra of all complex regular Borel measures (necessarily finite) on G with the total variation norm $\|\cdot\|$. Define

$$\begin{aligned} M(G,\omega) &:= \{\mu \in M_{loc}(G) : \omega\mu \in M(G)\} \\ M_0(G,\omega) &:= \{\mu \in M(G,\omega) : \widehat{\mu} \in C_0(\Delta(L^1(G,\omega)))\} \\ M_{00}(G,\omega) &:= \{\mu \in M(G,\omega) : \widehat{\mu} = 0 \text{ on } \Delta(M(G,\omega)) \setminus \Delta(L^1(G,\omega))\} \end{aligned}$$

Proposition 5.8 (13, Proposition 2.1)**.** *Let $\mu, \nu \in M_{loc}(G)$.*

(1) *$D_{\mathcal{L}\mu} \neq \emptyset$ iff $\mu \in M(G,\omega)$ for some weight ω on G.*
(2) *$D_{\mathcal{L}\mu} \cap D_{\mathcal{L}\nu} \neq \emptyset$ iff $\mu, \nu \in M(G,\omega)$ for some weight ω on G.*
In this case, $\mu * \nu \in M(G,\omega)$ and $D_{\mathcal{L}\mu*\nu} \neq \emptyset$.
(3) *If $\alpha \in D_{\mathcal{L}\mu}$, then $\alpha + \widehat{G} \subseteq D_{\mathcal{L}\mu}$.*
(4) *If $\alpha \in D_{\mathcal{L}\mu}$ and if $\mathcal{L}\mu = 0$ on $\alpha + \widehat{G}$, then $\mu = 0$.*
(5) *Let ω be a weight on G and let $\mu \in M(G,\omega)$. Then $H(G,\omega) \subseteq D_{\mathcal{L}\mu}$; the restriction of $\mathcal{L}\mu$ on $H(G,\omega)$ is the* generalized Fourier-Stieltjes transform *of μ.*
(6) *Let ω be a weight on G and let $f \in l1gw$. Then $H(G,\omega) \subseteq D_{\mathcal{L}f}$; the restriction of $\mathcal{L}f$ on $H(G,\omega)$ is the* generalized Fourier transform *of f.*
(7) *Let ω be a weight on G such that $\omega \geq 1$ on G and let $\mu \in M(G,\omega)$. Then $\widehat{G} \subseteq D_{\mathcal{L}\mu}$; the restriction of $\mathcal{L}\mu$ on $\widehat{G}$ is the* Fourier-Stieltjes transform *of μ.*
(8) *Let ω be a weight on G with $\omega \geq 1$ on G and let $f \in l1gw$. Then $\widehat{G} \subseteq D_{\mathcal{L}f}$; the restriction of $\mathcal{L}f$ on $\widehat{G}$ is the* Fourier transform *of f.*

Proposition 5.9 (13, Proposition 2.3)**.** *The following are equivalent:*
(i) $M(G,\omega)$ is regular; (ii) $M(G,\omega)$ has UUNP; (iii) $L^1(G,\omega)$ has UUNP and G is discrete.

Theorem 5.10 (13, Theorem 2.4)**.** *$M(G,\omega)$ has a minimum uniform norm if and only if $L^1(G,\omega)$ has UUNP.*

Proposition 5.11 (13, Proposition 2.6)**.** *The following are equivalent:*
(i) $M_{00}(G,\omega)$ is regular; (ii) $M_{00}(G,\omega)$ has UUNP; (iii) $L^1(G,\omega)$ has UUNP.

Conjecture: Motivated by the fact that the algebra $M_o(G)$ fails to have UUNP (7, Page-234), we conjecture that $M_0(G,\omega)$ has UUNP if and only if $L^1(G,\omega)$ has UUNP and G is discrete.

6. The Weighted Discrete Semigroup Algebra $\ell^1(S,\omega)$

The discrete analogue of the Beurling algebra $L^1(G,\omega)$ is the weighted discrete semigroup algebra $\ell^1(S,\omega)$. Given an abelian semigroup S, Hewitt and Zuckerman showed in [HZ] that the commutative convolution Banach algebra $\ell^1(S)$ is semisimple iff the set of all bounded semicharacters on S separates the points of S iff S has P_0-property: for $s, t \in S$, if $s^2 = t^2 = st$, then $s = t$; such semigroups were called *separating semigroups* in [HZ]. When S is a $*$-semigroup, the algebra $\ell^1(S)$ is a Banach $*$-algebra with the involution $f^*(s) = \overline{f(s^*)}$ $(s \in S)$. We search for an analogue of Hewitt-Zuckerman result for $*$-semisimplicy of $\ell^1(S)$. Though $*$-semisimplicity is equivalent to bounded hermitian semicharacters separating the points of S, a search for an analogue of the intrinsic P_0-property turns out to be difficult. This leads to several closely related separation properties. Some of which are necessary but not sufficient; and others are sufficient but not necessary. Let S be an abelian semigroup. A *bounded semicharacter* on S is a non-zero map $\alpha : S \to \mathbb{C}$ such that $|\alpha(s)| \leq 1$ and $\alpha(st) = \alpha(s)\alpha(t)$ $(s, t \in S)$. Let

$$\begin{aligned}
\Phi_{bs}(S) &:= \text{the set of all bounded semicharacters on } S; \\
\Phi_s(S) &:= \{\alpha \in \Phi_{bs}(S) : |\alpha(s)| = 0 \text{ or } 1 \ (s \in S)\}; \\
\Psi_{bs}(S) &:= \{\alpha \in \Phi_{bs}(S) : \alpha(s^*) = \overline{\alpha(s)} \ (s \in S)\}; \\
\Psi_s(S) &:= \Phi_s(S) \cap \Psi_{bs}(S).
\end{aligned}$$

We search for an involutive analogue of [HZ, Theorems 3.5, 5.6, 5.8]. We propose the following P_1-property: for $s, t \in S$, if $s^*s = t^*t = s^*t$, then $s = t$. We prove the following, which exhibits the intricacies involved, showing that the complete analogue is not true.

Theorem 6.1 (16, Theorem 2.2)**.** *Consider following statements for abelian $*$-semigroup S.*

(1) *$\ell^1(S)$ is $*$-semisimple.*

(2) *$\Psi_{bs}(S)$ separates the points of S.*

(3) *$\Psi_s(S)$ separates the points of S.*

(4) *S has P_1-property.*

Then (1) $\Leftrightarrow$ (2) $\Leftarrow$ (3)*,* (2) $\Rightarrow$ (4)*,* (2) $\not\Rightarrow$ (3)*, and* (4) $\not\Rightarrow$ (2)*.*

A $*$-semigroup S is *$*$-idempotent* if s^*s is idempotent for all $s \in S$ and if, for every $s \in S$, there exists $e_s \in S$ such that $s = e_s s^* = s^* e_s$. For example, every abelian group G with $g^* = g^{-1}$ $(g \in G)$ is a $*$-idempotent semigroup. For this class of $*$-semigroups, the following theorem implies that the semisimplicity and the $*$-semisimplicity are equivalent.

Theorem 6.2 (16, Theorem 2.3)**.** *Let S be a $*$-idempotent, abelian $*$-semigroup. Then $\ell^1(S)$ is semisimple iff it is $*$-semisimple.*

Theorem 2.18 above shows that the natural condition P_1 is not the correct involutive analogue of the condition P_0 so as to be equivalent to the $*$-semisimplicity. Our search for a correct intrinsic condition leads to the following conditions. Though experimental, they seem to be of some relevance.

Definition 6.3 (16, Definition 3.1). *Consider the following properties on a $*$-semigroup S.*

(1) $P_1 : s = t$ *whenever* $s^*s = t^*t = s^*t$.

(2) $P_2 : s = t$ *whenever* $ss^*s = ss^*t = tt^*t = st^*t$.

(3) $P_3 : s = t$ *whenever* $ss^*s = tt^*t = s^3 = t^3$.

(4) $P_4 : s = t$ *whenever* $ss^*s = tt^*t = s^2t = t^2s$.

(5) $P_5 : s = t$ *whenever* $ss^*s = tt^*t$.

(6) $Q_1 : ss^*s = s \ (s \in S)$.

(7) $Q_2 : s = t$ *whenever* $s^*t = t^*s$.

(8) $Q_3 : s = t$ *whenever* $s^*ts = st^*s$.

Proposition 6.4 (16, Proposition 3.2). *Let S be an abelian $*$-semigroup. Then*

(1) $P_2 \Leftrightarrow P_1 \Rightarrow P_0$.

(2) $P_5 \Rightarrow P_3$ *and* $P_5 \Rightarrow P_4$.

(3) $Q_3 \Leftrightarrow Q_2 \Rightarrow Q_1$.

Theorem 6.5 (16, Theorem 3.3). *Let S be an abelian $*$-semigroup. If $\ell^1(S)$ is $*$-semisimple, then S has P_i-property* $(i = 0, 1, 2, 3, 4, 5)$.

Theorem 6.6 (16, Theorem 3.4). *Let S be an abelian $*$-semigroup. If S has Q_1-property, then $\ell^1(S)$ is $*$-semisimple.*

7. Weighted Analogues of Theorems of Wiener and others

Wiener's classical theorem in Fourier series is the following. Let f be a continuous function on the unit circle Γ. Let f have absolutely convergent Fourier series. Then the celebrated Wiener's theorem [Ka, Theorem 2.2.11] states that if $f(z) \neq 0$ for all $z \in \Gamma$, then $1/f$ has absolutely convergent Fourier series. Then Lévy, Żelazko, and Domar generalized this result. We proved weighted analogues of the theorems of Wiener and Lévy. Let $\omega : \mathbb{Z} \longrightarrow [1, \infty)$ be a weight on $\mathbb{Z}$. A function $f \in C(\Gamma)$ has *ω-absolutely convergent Fourier series* if $\sum_{n \in \mathbb{Z}} |\widehat{f}(n)| \omega(n) < \infty$.

Theorem 7.1 (10). *Let $f \in C(\Gamma)$ have ω-absolutely convergent Fourier series.*

(I) *If $f(z) \neq 0$ $(z \in \Gamma)$, then there exists a weight ν on $\mathbb{Z}$ such that*

(1) *$1/f$ has ν-absolutely convergent Fourier series;*

(2) *ν is non-constant if and only if ω is non-constant;*

(3) $\nu(n) \leq \omega(n)$ $(n \in \mathbb{Z})$.

(II) *If φ is a holomorphic function on some neighbourhood of the range of f, then there exists a weight χ on $\mathbb{Z}$ such that*

(1) *$\varphi \circ f$ has χ-absolutely convergent Fourier series;*

(2) *χ is non-constant if and only if ω is non-constant;*

(3) $\chi(n) \leq \omega(n)$ $(n \in \mathbb{Z})$.

Then both of us together with Dr. Dabhi proved weighted analogues of Żelazko and Domar. Let $0 < p \leq 1$. A continuous function $f \in C(\Gamma)$ has *p^{th}-power ω-absolutely convergent Fourier series* if $\sum_{n\in\mathbb{Z}} |\widehat{f}(n)|^p \omega(n) < \infty$.

Theorem 7.2 (14, Theorem 1)**.** *Let $0 < p \leq 1$, let ω be a weight on $\mathbb{Z}$, and let $f \in C(\Gamma)$ have p^{th} power ω-absolutely convergent Fourier series.*

(I) *If $f(z) \neq 0$ $(z \in \Gamma)$, then there exists a weight ν on $\mathbb{Z}$ such that*

(1) *$1/f$ has p^{th} power ν-absolutely convergent Fourier series;*

(2) *ν is non-constant if and only if ω is non-constant;*

(3) *$\nu(n) \leq \omega(n)$ $(n \in \mathbb{Z})$.*

(II) *If φ is a holomorphic function on some neighbourhood of the range of f, then there exists a weight χ on $\mathbb{Z}$ such that*

(1) *$\varphi \circ f$ has p^{th} power χ-absolutely convergent Fourier series;*

(2) *χ is non-constant if and only if ω is non-constant;*

(3) *$\chi(n) \leq \omega(n)$ $(n \in \mathbb{Z})$.*

A weight $\omega \geq 1$ on $\mathbb{Z}$ is *non-quasi analytic* if $\sum_{n\in\mathbb{Z}} \frac{\log \omega(n)}{1+n^2} < \infty$. Domar proved in [Do, Theorem 2.11] that if f has ω-absolutely convergent Fourier series and is nowhere vanishing on Γ, then $1/f$ has ω-absolutely convergent Fourier series provided ω is non-quasi analytic. This result is a special case of the following corollary which follows from above theorem.

Corollary 7.3. *Let $0 < p \leq 1$, let ω be a non-quasi analytic weight on $\mathbb{Z}$, and let $f \in C(\Gamma)$ be nowhere vanishing. If f has p^{th} power ω-absolulely convergent Fourier series, then $1/f$ has p^{th} power ω-absolutely convergent Fourier series.*

Analogous to Gel'fand's proof of Wiener theorem [GRS, Page-33], which is based on Banach algebras, we shall use Gel'fand theory of p-Banach algebras developed by Żelazko in the frame work of locally bounded Beurling algebras. Let $\mathcal{A}$ be a (complex) algebra and let $0 < p \leq 1$. Then a mapping $\|\cdot\| : \mathcal{A} \longrightarrow \mathbb{R}$ is a *p-norm* on $\mathcal{A}$ if, for $x, y \in \mathcal{A}$ and for $\alpha \in \mathbb{C}$,

(1) $\|x\| \geq 0$ and $\|x\| = 0$ if and only if $x = 0$;

(2) $\|x + y\| \leq \|x\| + \|y\|$;

(3) $\|\alpha x\| = |\alpha|^p \|x\|$;

(4) $\|xy\| \leq \|x\|\|y\|$.

If $\mathcal{A}$ is complete in the p-norm, then $(\mathcal{A}, \|\cdot\|)$ is a *p-Banach algebra*. Among unital algebras, p-Banach algebras are precisely the complete locally bounded algebras [Z, Theorem 2.3]. Given a continuous weight ω on a locally compact group G satisfying $\omega(st) \leq \omega(s)\omega(t)$ $(s, t \in G)$, let

$L^p(G, \omega)$ be the set of all measurable functions $f : G \longrightarrow \mathbb{C}$ such that

$$\|f\|_{p,\omega} := \int_G |f(t)|^p \omega(t)\, dm(t) = \int_G |f(t)|^p\, dm_\omega(t) < \infty.$$

For a discrete abelian group G and for $0 < p \leq 1$, the space $\ell^p(G)$ is a p-Banach algebra with convolution. In fact, Żelazko proved that, for a locally compact group G and for $0 < p < 1$, the complete locally bounded space $L^p(G)$ is closed under convolution if and only if G is discrete. The following theorem gives a Beurling algebra analogue of this.

Theorem 7.4 (14, Theorem 2)**.** *Let $0 < p < 1$, let G be a non-compact, locally compact group, and let ω be a continuous weight on G. Then the following are equivalent.*

(1) *$L^p(G, \omega)$ is closed under convolution.*
(2) *G is discrete.*
(3) *$L^p(G, \omega)$ admits a non-zero continuous linear functional.*
(4) *The set of continuous linear functionals on $L^p(G, \omega)$ separates the points of $L^p(G, \omega)$.*
(5) *G admits an atom.*
(6) *G admits sufficiently many atoms.*

In this case, if G is abelian or ω is symmetric, then $L^p(G, \omega)$ is semisimple.

8. Open Problems

We believe that the following problems are still open.

(1) Let A be a semisimple, commutative Banach algebra. Assume that A has UUNP. Does A have SEP?
(2) Let G be an LCA group and ω be a weight on G. Is it true that $M_0(G, \omega)$ has UUNP if and only if $L^1(G, \omega)$ has UUNP and G is discrete?
(3) Let G be a non-abelian, locally compact group and ω be a weight on G. Is $L^1(G, \omega)$ always semisimple?
(4) Let S be an abelian $*$-semigroup. Is it possible to characterize the $*$-semisimplicity of the algebra $\ell^1(S)$ in terms of the abelian $*$-semigroup S?
(5) Let S be an infinite semigroup. Does there always exist an unbounded weight on S?

My Research Papers with Professor Bhatt

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