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WELL-BEHAVED *-REPRESENTATIONS AND AUTOMATIC CONTINUITY IN LOCALLY CONVEX *-ALGEBRAS

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1. Tribute

I happened to know about Professor S.J. Bhatt, of Sardar Patel University, Vallabh Vidyanagar, from my first steps as a researcher, since both of us were working in the same field of Mathematics. So we always were exchanging our research work and we were using the results of each another. It was only in 2005 that I had the honour to meet him for the first time, in the fifth International Conference on "Topological Algebras and Applications", Athens (Greece). I would have seen him again in 2018, in Gujarat, in the "International Conference on Banach Algebras, Harmonic Analysis, and Operator Theory", but unfortunately because of some health problems, I missed the opportunity to see Subhash, once more. Both of us had already started collaborating, together with Professor A. Inoue of Fukuoka University, Japan, since 2003. As a result the three of us have published three papers [1, 2, 3]. A fourth joint paper [4], this time with A. Inoue and D.J. Karia, was published in 2007. I would like to add that a lot of distinguished results of Professor S.J. Bhatt (as the only author, or in collaboration) have been included in my book [FRA2], published, in 2005. The contribution of his research work to the mathematical community is highly appreciated and it has been used by numerous researchers.

2. Our Joint Research

My joint work with S. J. Bhatt and A. Inoue in [1, 2, 3] is mainly referred to the study and structure of the so-called *well-behaved *-representations on locally convex *-algebras.* Our last collaborative work in [4], this time joint with A. Inoue and D. J. Karia, concerns various topics on locally convex *-algebras connected with *-representations. Particular attention has been given to the investigation of *properties that a unital complete locally convex *-algebra may have, in order to admit a C*-enveloping algebra.* (C*-)enveloping algebras play a fundamental role in the *-representation theory of Banach and non-normed topological *-algebras (cf. e.g., [FRA2, Chapter IV]).

We start discussing the main results of [1, 2, 3]. Note that we always deal with complex associative algebras and Hausdorff topological spaces.

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It is well known that C*-seminorms are related to bounded *-representations, factoring through their null space. It is also well known that *unbounded *-representations* are those corresponding to unbounded linear operators, meaning that their (common) domain is a dense subspace of a Hilbert space H and not the whole H. *The unbounded operators may sometimes show pathological behaviour, so naturally one tries to find ways to pick up among all unbounded *-representations of a given *-algebra, the best possible ones.* In this way the *well-behaved *-representations* appeared on the scene several years ago by S.J. Bhatt, A. Inoue and H. Ogi [BIO, p. 54] (see also [BIK, p. 420]) and indepedently by K. Schmüdgen in [SCH]. The relationship between the two approaches is investigated in [BIK, Subsection 6.1], where concrete examples of well-behaved *-representations are discussed too; for more examples, see also [2, Examples 3.9], as well [3, Section 5]. A major role for the construction of such *-representations is played by the so-called *unbounded C*-seminorms* [BIO, BIK]. The last terminology, is justified from the fact that *this kind of C*-seminorms are defined not on the whole, but on a *-subalgebra of a given *-algebra* and they give rise to unbounded *-representations. Similarly, an *unbounded m*-seminorm* is defined, where by an *m*-seminorm* we mean a submultiplicative *-seminorm. In Distribution Theory and Quantum Field Theory there are examples of locally convex *-algebras that admit unbounded C*-seminorms. For unbounded m*-seminorms, see [3, Example 5.2, (1), (2), (3)]. The simplest example of an unbounded C*-seminorm is given by considering the *-algebra $A[\tau] \equiv C[0,1]$ of all continuous complex-valued functions on $[0,1]$, under the topology τ induced by the *-seminorms

$$p_n(f) := \left[\int_0^1 \mid f(x) \mid^n dx\right]^{1/n}, \ f \in C[0,1], \ n = 1, 2, \ldots$$

Let $D(p_\infty) := \{f \in A[\tau] : \sup_n p_n(f) < \infty\}$ with $p_\infty(f) := \sup_n p_n(f)$, $f \in D(p_\infty)$. Then $D(p_\infty)$ is a *-subalgebra of $A[\tau]$ and $p_\infty(f) = \|f\|_\infty$, for each $f \in A[\tau]$. In particular, p_∞ is an unbounded C*-(semi)norm of $A[\tau]$ and $D(p_\infty) = C[0,1][\|\cdot\|_\infty] = I_{p_\infty}$; so that $I_{p_\infty} \neq \{0\} = \ker p_\infty$.

Given an unbounded C*-seminorm p on a *-algebra A, denote its domain by $D(p)$. The set $\ker p \equiv \{x \in D(p) : p(x) = 0\}$ is a *-ideal of $D(p)$ and $I_p \equiv \{x \in D(p) : ax \in D(p) \ \forall \ a \in A\}$ is the largest left ideal of A contained in $D(p)$. *The position of the left ideal I_p in relation to the *-ideal* $\ker p$ *determines the existence of an unbounded *-representation derived by the unbounded C*-seminorm* p (see [1, pp. 125-127]).

• Construction of an unbounded *-representation from a C*-seminorm p on a *-algebra A, such that $I_p \not\subseteq \ker p$ (cf. [BIO]).

The Hausdorff completion of $D(p)[p]$ ($D(p)$ endowed with the induced by p topology), as a C*-algebra admits a faithful nondegenerate *-representation Π_p on a Hilbert space H_{Π_p}. Define

$$\pi_p^\circ : D(p) \to B(H_{\pi_p}) \text{ with } \pi_p^\circ(x) := \Pi_p(x + \ker p),\ \forall\ x \in D(p).$$

Then, π_p° is a bounded *-representation on $D(p)[p]$. Put now

$$D(\pi_p) := [\Pi_p(x + \ker p)\xi : x \in I_p,\ \xi \in H_{\Pi_p}] \subseteq H_{\Pi_p}.$$

For $a \in A$ and $\sum_{\text{finite}} \Pi_p(x_\kappa + \ker p)\xi_\kappa \in D(\pi_p)$, define the map

$$\pi_p(a)\left(\sum_\kappa \Pi_p(x_\kappa + \ker p)\xi_\kappa\right) := \sum_\kappa \Pi_p(ax_\kappa + \ker p)\xi_\kappa.$$

This is a well defined linear operator on $D(\pi_p)$. Taking the Hilbert space H_{π_p} given by the norm-closure of $D(\pi_p)$ in H_{Π_p}, we obtain a non-trivial unbounded *-representation π_p of A in H_{π_p}. The relation $I_p \not\subseteq \ker p$ equivalently means that $H_{\pi_p} \neq \{0\}$.

• Selection of well-behaved *-representations among all unbounded *-representations derived by unbounded C^*-seminorms.

The unbounded *-representation π_p as before is *well-behaved*, whenever $H_{\pi_p} = H_{\Pi_p}$. In the example we mentioned above, with the unbounded C*-(semi)norm p_∞, the corresponding unbounded *-representation π_{p_∞} of $A[\tau]$ is well-behaved. The *-algebra $A[\tau]$ is a metrizable locally convex *-algebra, the completion $\widetilde{A}[\tau]$ of which is the Arens algebra $L^w[0,1] := \bigcap_{1 \le p < \infty} L^p[0,1]$. In this case, the corresponding ideal $I_{\widetilde{p}_\infty}$ is trivial, so that we can not have an unbounded well-behaved *-representation of $\widetilde{A}[\tau]$. Below, we present several ways of selection of well-behaved *-representations, given by the main results of [2], from where the terminology and notation used, can be found.

Theorem 2.1. [2, Theorem 3.5] *Let p be an unbounded m*-seminorm on a *-algebra A, such that $I_p \neq \{0\}$. Then, the following statements are equivalent:*

(i) *The unbounded m*-seminorm p is strongly representable.*

(ii) *A p-continuous representable positive linear functional f exists on $D(p)$ with $I_p \not\subset \ker f$.*

(iii) *A well-behaved *-representation π_r exists on A, induced by a w-semifinite unbounded C*-seminorm r of A, with $D(r) = D(p)$, $r \le p$ and $I_r \not\subset \ker r$.*

Theorem 2.2. [2, Theorem 4.5] *Let A be a pseudo-complete locally convex *-algebra with an identity element. Then the following statements are equivalent:*

(i) *There exists a well-behaved *-representation of A derived by a tw-semifinite unbounded C*-seminorm r of A with $D(r) = A_b$* (the latter is the *-subalgebra of A generated by the self-adjoint (Allan-)bounded elements of A).

(ii) *There exists a nontrivial uniformly nondegenerate *-representation π of A.*

(iii) *There exists an unbounded C*-seminorm* r *on* A *with* $D(r) = A_b$ *and* $I_b \not\subset \ker r$.

(iv) *There exists a nonzero* (positive) *linear functional* f *on* I_b *with* $|f(x)|^2 \leq f(x^*x)$, *for each* $x \in I_b$.

Proposition 2.3. [2, Propositions 4.7, 4.8] (1) *Let* $A[\tau]$ *be a Fréchet locally convex *-algebra with a uniformly left bounded approximate identity. Let* π *be a nontrivial unbounded* $*$*-representation of* A *and* I *a left ideal of* A *contained in* A_b, *such that* $[\pi(I)D(\pi)] = H_\pi$. *Then,* $\pi : A[\tau] \to \pi(A)[\tau_{lu}^I]$ *is continuous.*

(2) *Let* $A[\tau]$ *be a quasi-complete bornological GB*-algebra and* π *a uniformly nondegenerate *-representation of* A. *Then,* $\pi : A[\tau] \to \pi(A)[\tau_{lu}]$, *is continuous.*

Note that *GB*-algebras are unbounded generalizations of C*-algebras initiated by G.R. Allan, in 1967. The PhD thesis of S.J. Bhatt was a study on these unbounded operator algebras.*

Concerning the existence of faithful well-behaved *-representations, we have the following

Theorem 2.4. [2, Theorem 5.3, Corollary 5.4] *Let* $A[\tau]$ *be a GB*-algebra. Suppose that the Pedersen ideal* K_{A_b} *of the C*-algebra* A_b *is contained in* I_b. *Then,* $A[\tau]$ *admits a faithful well-behaved *-representation.*

For the last assumption in Theorem 2.4, see also [YNG].

We exhibit now the main results of [3], to which the reader is referred for terminology and notation. In this paper we continue the study of certain well behaved *-representations called *spectral* (resp. *p-spectral*) induced from some unbounded m*-seminorm p. The existence of such well-behaved *-representations is investigated in terms of certain properties of the well known Pták function that plays an important role in the theory of hermitian Banach *-algebras developed by V. Pták, in 1972 [PT] (see also [FRA2, Chapter V, Section 22]). For a *-algebra A, with spectral radius r_A, finite for all self-adjoint elements of A, the Pták function p_A is given as follows: $p_A(x) := r_A(x^*x)^{1/2}$, for each $x \in A$.

Theorem 2.5. [3, Theorem 3.1] *Let* p *be an unbounded m*-seminorm on a *-algebra* A *with an identity element. The following statements are equivalent:*

(i) *There exists a p-spectral well behaved *-representation* π_r *of* A *induced by a w-semifinite unbounded C*-seminorm* r *of* A, *with* $D(r) = D(p)$.

(ii) *There exists a p-spectral *-representation* π *of* A, *such that* $[\pi(I_p)H_\pi] = H_\pi$.

(iii) *The Pták function* $p_{D(p)}$ *is a w-semifinite C*-seminorm on* $D(p)$.

Proposition 2.6. [3, Theorem 3.5, Corollary 3.7] *On a given Fréchet locally convex *-algebra* $A[\tau]$, *with* τ *defined by an increasing family* $\{p_n\}_{n\in\mathbb{N}}$ *of *-seminorms, the following statements are equivalent:*

(i) *There exist a well behaved *-representation* π_r *of* A *induced by a w-semifinite unbounded C*-seminorm* r *of* A*, such that* $D(r) = D(p_\infty) := \{x \in A : p_\infty(x) \equiv \sup_{n\in\mathbb{N}} p_n(x) < \infty\}$ *and a spectral unbounded C*-seminorm* r' *of* A*, such that* $D(r') = D(p_\infty)$ *and* $\|\overline{\pi_r(x)}\| \leq r'(x) \leq p_\infty(x),\ x \in D(p_\infty)$.

(ii) *The Pták function* $p_{D(p_\infty)}$ *is a C*-seminorm on* $D(p_\infty)$*, with* $p_\infty(I_\infty) \neq \{0\}$.

(iii) *The Banach *-algebra* $D(p_\infty)$ *is hermitian and* $p_\infty(I_\infty) \neq \{0\}$.

The following theorem is a version of Theorem 2.5 in the case of locally convex *-algebras, combined with [2, Theorem 4.5].

Theorem 2.7. [3, Theorem 4.4] *Let* A *be a pseudo-complete locally convex *-algebra with an identity element. The following statements are equivalent:*

(i) *There exist a well behaved *-representation* π_r *of* A *induced by a tw-semifinite unbounded C*-seminorm* r *of* A*, with* $D(r) = A_b$ *and a spectral* (in the sense that $r_{A_b}(x) \leq r'(x)$, for all $x \in A_b$) *unbounded C*-seminorm* r' *of* A*, with* $D(r') = A_b$ *and* $\|\overline{\pi_r(x)}\| \leq r'(x),\ x \in A_b$.

(ii) *There exist a tw-semifinite unbounded C*-seminorm* r *of* A*, with* $D(r) = A_b$ *and a spectral unbounded C*-seminorm* r' *of* A*, with* $D(r') = A_b$ *and* $r(x) \leq r'(x),\ x \in A_b$.

(iii) *The Pták function* p_{A_b} *is a C*-seminorm on* A_b *with* $p_{A_b}(I_b) \neq \{0\}$.

In [3, Section 5], the reader can find interesting examples on the construction of spectral, as well p-spectral well behaved *-representations in various classes of locally convex *-algebras. In particular, taking the Fréchet algebra $C^\infty([0,1], A)$, $A = M_n(\mathbb{C})$, $n > 1$, of all A-valued smooth functions on $[0,1]$ as in [3, Example 5.4], one concludes that the Pták functions $p_{C^\infty[0,1]}$ and p_A are C*-seminorms with the property $I_{C^\infty[0,1]} = \{0\} = I_{p_A}$, which implies that no well behaved unbounded *-representations can be induced on $C^\infty([0,1], A)$, in terms of its hermitian *-subalgebras $C^\infty[0,1]$ and A. Thus, the following **question** arises: *does* $C^\infty([0,1], A)$ *admit any kind of unbounded *-representations?* Notice that considering a locally compact (noncompact) space X and a non-unital hermitian Banach *-algebra A, then the *-algebra $C(X, A)$ of all A-valued continuous functions on X under the topology of uniform convergence on the compacts of X, does admit well behaved unbounded *-representations [3, Example 5.3].

All the preceding results have been used, on the one hand for the construction of well behaved unbounded *-representations on tensor product locally convex *-algebras [FRI] and on the other hand for providing applications in tensor product GB*-algebras [FIW, Section 7].

We continue with the main results of [4]. There is an abundance of non-normed topological *-algebras that admit a C*-enveloping algebra (see [B, BK]). Characterizations of such algebras are given in terms of various properties of decisive features of a given topological *-algebra A, like the set $R_c(A)$ of its nontrivial continuous *-representations or the set $P_c(A)$ of its continuous positive functionals, the hermitian spectrum $sp_A^h(\cdot)$, or the hermitian Pták function $p_A^h(\cdot)$, etc. There are also interrelations of this sort of algebras with hermiticity and C*-spectrality (for the last term, see e.g. [BIO]). In this regard, we have the following

Theorem 2.8. [4, Theorem 2.6] *Let $A[\tau]$ be a complete locally convex *-algebra, which either fulfills the condition $r_A \restriction_{H(A)} < \infty$, or has a continuous inversion and each of its elements is* (Allan-)*bounded. Then, the following hold:*

(1) *$A[\tau]$ has a C*-enveloping algebra $\Leftrightarrow$ the set $R_c(A)$ is nonempty and there exists a continuous seminorm p on A, such that $p_A^h(x) \leq p(x)$, for all $x \in A$, $\Leftrightarrow$ there is $\pi \in R_c(A)$, such that $p_A^h(x) = \|\pi(x)\|$, for all $x \in A$. In particular, if $A[\tau]$ is Fréchet, then $A[\tau]$ has a C*-enveloping algebra $\Leftrightarrow R(A)(= R_c(A)) \neq \emptyset$.*

(2) *$A[\tau]$ is a C*-spectral algebra $\Leftrightarrow$ $A[\tau]$ has a C*-enveloping algebra and p_A is a continuous C*-seminorm. In this case, $p_A = p_A^h$. In particular, if $A[\tau]$ is Fréchet, then $A[\tau]$ is C*-spectral $\Leftrightarrow$ $A[\tau]$ has a C*-enveloping algebra and p_A is a C*-seminorm on $A[\tau]$ $\Leftrightarrow$ $A[\tau]$ is hermitian with a C*-enveloping algebra.*

Theorem 2.9. [4, Theorem 2.7] *Let $A[\tau]$ be a complete locally convex *-algebra with an identity element. Then, the following are equivalent:*

(1) *$A[\tau]$ has a C*-enveloping algebra.*

(2) *$P_c(A)$ is equicontinuous.*

(3) *$R_c(A)$ is equicontinuous.*

(4) 0 *is an interior point of the set K of all $x \in A$, such that $\pi(x)$ is quasi-invertible in $B(H_\pi)$, for all $\pi \in R_c(A)$.*

(5) *There exists a continuous *-seminorm p on A, such that $r_A^h(x) \leq p(x)$, for all $x \in A$.*

(6) *There exists a continuous *-seminorm p on A, such that ${p_A}'^h(x) \leq p_A^h(x) \leq p(x)$, for all $x \in A$, where ${p_A}'^h(\cdot)$ is the corresponding Pták function defined through $r_A'^h(\cdot)$* (as in (5′), below).

If $A[\tau]$ is also Fréchet, then each of the above is equivalent to

(7) *${sp_A}^h(x)$ is bounded for all $x \in A$.*

If $A[\tau]$ is m-convex, then each of* (1) - (6) *is equivalent to*

(2′) *The set $B_c(A)$ of the nonzero extreme points of $P_c(A)$ is equicontinuous.*

(3′) *The set ${R_c}'(A) = \{\pi \in R_c(A) : \pi$ is topologically irreducible$\}$ is equicontinuous.*

(4′) 0 *is an interior point of K'.*

(5′) *There exists a *-seminorm p in A, from those defining τ, such that $r_A'^h(x) \leq p(x)$, for all $x \in A$.*

If $A[\tau]$ is Fréchet and m-convex, then each of* (2′) - (5′) *is equivalent to*

(7′) *${sp_A}'^h(x)$ is bounded for all $x \in A$.*

Note that 'dash' in the enumeration $(2')$ - $(5')$ and $(7')$, as well in the analogous sets as in (2) - (5) and (7) means that these sets are now defined through ${R_c}'(A)$ instead of $R_c(A)$.

A structural analogy between (not necessarily involutive) commutative Q-algebras and topological *-algebras having a C*-enveloping algebra, leads to the study of some automatic continuity results. First we state the following

Conjecture 2.10. [4, Conjecrure 3.1] *Let $\varphi : A[\tau_A] \to B[\tau_B]$ be an injective *-homomorphism from a pro-C*-algebra into a complete m*-convex algebra $B[\tau_B]$, such that $\overline{Im\varphi}$ has a C*-enveloping algebra. Then, ${\varphi^{-1}}_{|\overline{Im\varphi}}$ is continuous, and $Im\varphi$ is closed in $B[\tau_B]$.*

Theorem 2.11 that follows, supports Conjecture 2.10 and also provides an improvement to [FRA1, Theorem 3.9].

Theorem 2.11. [4, Theorem 3.2] *Let $A[\tau]$ be a pro-C*-algebra with an identity element; let $A'[\tau']$ be a complete m*-convex algebra and $\varphi : A[\tau] \to A'[\tau']$ an injective *-homomorphism, such that $B = \overline{Im\varphi}$ (with the relative topology) has a C*-enveloping algebra. Assume that one of the following conditions holds:*

(1) *A is commutative.*

(2) *B is *-semisimple.*

(3) *B is Fréchet and pseudo Q-algebra.*

(4) *φ is continuous.*

Then, ${\varphi^{-1}}_{|\overline{Im\varphi}}$ is continuous, $Im\varphi = \overline{Im\varphi}$, $Im\varphi$ is a semisimple Q-algebra and $A = A_b$. Further, if φ is continuous, then the topology τ of A is normable.

Notice that for the proof of Theorem 2.11, a series of lemmas (providing interesting information) is needed (see [4, pp. 81, 82]). The conjecture that follows is motivated by the fact that a *-homomorphism from an m*-convex Q-algebra to a C*-algebra is continuous [FRA1, Theorem 3.1] and that a homomorphism from a Q-algebra to a uniform Banach algebra is continuous.

Conjecture 2.12. [4, Conjecture 3.7] *Let $A[\tau]$ be a complete m*-convex algebra with a C*-enveloping algebra. Let π be a *-homomorphism of $A[\tau]$ into a C*-algebra $B[\|\cdot\|]$. Then, π is continuous.*

From mid 90's of the last century, considerable attention has been given to the structure of locally convex *-algebras (resp. m*-convex algebras) regarded as dense *-subalgebras of C*-algebras. Section 4 of [4] is devoted to such a study. Among others, we have the following

Theorem 2.13. [4, Theorem 4.6] *Let $(A_n[\|\cdot\|_n)$ be a direct increasing sequence of C^*-algebras. Put $B \equiv \bigcup_{n=1}^{\infty} A_n$ and let $A := \varinjlim_n A_n = C^* - \varinjlim_n A_n$ be the C^*-algebra direct limit. Let τ be the* *finest locally convex* (linear) *topology on* B *making each of the embeddings* $id_n : A_n[\|\cdot\|_n] \to B[\tau]$ *continuous. Then, the following hold:*

(1) $B[\tau]$ *is a complete m*-convex algebra continuously and densely embedded in* A*.*

(2) $B[\tau]$ *is spectrally invariant in* A*, in the sense that* $sp_B(x) = sp_A(x), \forall\, x \in A$*.*

(3) $B[\tau]$ *is a* Q*-algebra, hence it has a C*-enveloping algebra given by* A*; the C*-norm on* A *is a greatest* τ*-continuous C*-seminorm on* B*.*

Remark 2.14. In [4, p. 88] interesting examples are presented concerning the **question:** *when a dense Fréchet *-subalgebra of a C*-algebra has a C*-enveloping algebra?*

My Research Papers with Professor Bhatt

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