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UNBOUNDED C^*-SPECTRAL ALGEBRAS AND DIFFERENTIAL STRUCTURES OF C^*-ALGEBRAS

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1. Tribute

We were interested in S. J. Bhatt's paper "Representability of positive linear functionals on abstract star algebras without identity with applications to locally convex $*$-algebras", Yokohama Math. J. 29 (1981), 7-16 and invited him to our university "the Department of Applied Mathematics in Fukuoka University" for about one month during September and October, 1994. Ever since our joint research projects about the structure and representation theory of locally convex $*$-algebras began [(1),(2)] and continued [(3)-(6),(8)]. Since he again visited our department during May and June, 2006, we proceeded with the study of differential structures in C^*-algebras [(7),(9)]. We are grateful that his extensive knowledge has allowed us to broaden our research.

2. Our Joint Research

We begin with the basic definitions and properties about locally convex $*$-algebras and unbounded $*$-representations of $*$-algebras used in all of our joint papers (1)-(9). A *locally convex $*$-algebra* is a $*$-algebra which is also a Hausdorff locally convex space which that the multiplication is separately continuous in both variables and the involution is continuous. A *locally m-convex $*$-algebra* (or a locally C^*-convex $*$-algebra) is a locally convex $*$-algebra defined by a family of submultiplicative m^*-seminorms (or C^*-seminorms), and in particular, a complete locally C^*-convex $*$-algebra is called a *pro-C^*-algebra*. Such algebras were first studied by Arens [Ar], Michael [Mi] and others [Fr, Ph, etc.]. It is known that every complete locally m-convex $*$-algebra (resp. pro-C^*-algebra) is a projective limit of Banach $*$-algebras (resp. C^*-algebras). However, it is difficult to study general locally convex $*$-algebras which are not m-convex even if the multiplication is jointly continuous. Allan [Al, 1] studied general locally convex $*$-algebras. In particular, he introduced the notion of a *GB^*-algebra*, which is a generalization of a C^*-algebra and showed that a commutative GB^*-algebra is isomorphic to a $*$-algebra of continuous

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functions on a compact space taking the valued ∞ on a nowhere dense subset [Al, 2]. Dixon [Di] showed that every non-commutative GB^*-algebra is isomorphic to an unbounded operator algebra called an *EC^*-algebra.* We here review these algebras. Let $\mathscr{A}$ be a locally convex $*$-algebra. An element x of $\mathscr{A}$ is called *(Allan) bounded* if there exists a $\lambda > 0$ such that $\{(\lambda^{-1}x)^n;\ n \in \mathbb{N}\}$ is bounded, and the radius $\beta(x)$ of boundedness of x is defined by $\inf\{\lambda > 0;\ \{(\lambda^{-1}x)^n\}$ is bounded$\}$. Let $\mathscr{A}_0$ be the *bounded part* of $\mathscr{A}$, consisting of all Allan bounded elements of $\mathscr{A}$. If $\mathscr{A}$ is commutative, then $\mathscr{A}_0$ is a $*$-subalgebra of $\mathscr{A}$, but it is not even a subspace of $\mathscr{A}$ in general. Hence we consider the $*$-subalgebra of $\mathscr{A}$ generated by $(\mathscr{A}_0)_h := \{x \in \mathscr{A}_0;\ x^* = x\}$ as the *bounded $*$-subalgebra* in $\mathscr{A}$, and denote by $\mathscr{A}_b$. By $\mathfrak{B}$ we denote the collection of all subsets B of $\mathscr{A}$ such that B is bounded, closed and absolutely convex, and $B^2 \subset B$. For any $B \in \mathfrak{B}$, let $\mathscr{A}[B]$ denote the subspace of $\mathscr{A}$ generated by B. Then $\mathscr{A}[B] = \{\lambda x;\ \lambda \in \mathbb{C},\ x \in \lambda B\}$ and the equation: $\|x\|_B := \inf\{\lambda > 0;\ x \in \lambda B\}$ defines a norm on $\mathscr{A}[B]$, which makes $\mathscr{A}[B]$ a normed algebra. If $\mathscr{A}[B]$ is complete for each $B \in \mathfrak{B}$, then $\mathscr{A}$ is said to be *pseudo-complete.* A pseudo-complete locally convex $*$-algebra $\mathscr{A}$ with identity is said to be a *GB^*-algebra* if $\mathfrak{B}^* := \{B \in \mathfrak{B};\ B^* = B\}$ has a greatest member B_0 and $(1 + x^*x)^{-1} \in \mathscr{A}[B_0]$ for each $x \in \mathscr{A}$. If $\mathscr{A}$ is a GB^*-algebra, then $\mathscr{A}[B_0]$ is a C^*-algebra with C^*-norm $\|\cdot\|_{B_0}$. Let $\mathcal{D}$ be a dense subspace in a Hilbert space $\mathcal{H}$ and $\mathcal{L}^\dagger(\mathcal{D})$ denote the set of all linear operators X in $\mathcal{H}$ with the domain $\mathcal{D}$ for which $X\mathcal{D} \subset \mathcal{D}$, $\mathcal{D}(X^*) \supset \mathcal{D}$ and $X^*\mathcal{D} \subset \mathcal{D}$. Then $\mathcal{L}^\dagger(\mathcal{D})$ is a $*$-algebra with the identity operator I under the usual linear operators and the involution $X \to X^\dagger := X^*\lceil_{\mathcal{D}}$. A $*$-subalgebra of the $*$-algebra $\mathcal{L}^\dagger(\mathcal{D})$ is said to be an *O^*-algebra* on $\mathcal{D}$ in $\mathcal{H}$. A *$*$-representation* π of a $*$-algebra $\mathscr{A}$ on a Hilbert space $\mathcal{H}$ with a domain $\mathcal{D}$ is a $*$-homomorphism of $\mathscr{A}$ into $\mathcal{L}^\dagger(\mathcal{D})$, and then we write $\mathcal{D}$ and $\mathcal{H}$ by $\mathcal{D}(\pi)$ and $\mathcal{H}_\pi$, respectively. Let π be a $*$-representation of $\mathscr{A}$. If $\mathcal{D}(\pi)$ is complete with respect to the graph topology t_π defined by the family of seminorms $\{\|\cdot\|_{\pi(x)} := \|\cdot\| + \|\pi(x)\cdot\|;\ x \in \mathscr{A}\}$, then π is said to be *closed.* It is well known that π is closed if and only if $\mathcal{D}(\pi) = \cap_{x\in\mathscr{A}} D(\overline{\pi(x)})$. The *closure* of π is defined by $\mathcal{D}(\tilde{\pi}) := \cap_{x\in\mathscr{A}} D(\overline{\pi(x)})$. Then $\tilde{\pi}$ is the smallest closed extension of π. Unbounded $*$-representations of $*$-algebras were considered for the first time in 1962, independently by H.J. Borchers [Bo] and A. Uhlmann [Uh] in the Wightman formulation of quantum field theory. A systematic study was undertaken only at the begining of 1970, first by R.T. Powers [Po] and G. Lassner [La], then by many mathematician, from the pure mathematical situations and the physical applications. A survey of the theory of unbounded $*$-representations may be found in the monograph of K. Schmüdgen [Sc, 1] and the lecture note of A.I. of us [In].

In (1) we have defined the notion of hereditary C^*-spectral algebras and characterized it. Y. Yood has studied C^*-seminorms on a $*$-algebra in [Yo], and according T.W. Palmer [Pa], spectral $*$-algebras, C^*-spectral algebras and spectral bounded $*$-representations have been defined. In this paper we have defined the notions of hereditary spectral $*$-algebras, hereditary

C^*-spectral algebras, hereditary bounded $*$-representations and stable $*$-algebras, and obtained the following result:

Theorem 1. [(1): Theorem 1.6]. *Let $\mathscr{A}$ be a $*$-algebra. Then the following are equivalent:*

(i) $\mathscr{A}$ is a hereditary C^*-spectral algebra.

(ii) $\mathscr{A}$ admits a hereditary spectral $*$-representation.

(iii) $\mathscr{A}$ is stable and is a spectral $*$-algebra.

Our paper (2) has dealt with the admissibility and approximately admissibility of (quasi-) weight on $*$-algebras. Let φ be a quasi-weight on $P(\mathcal{N}_\varphi)$ in a $*$-algebra, where $P(\mathcal{N}_\varphi)$ is a positive cone generated by a left ideal $\mathcal{N}_\varphi$ of $\mathscr{A}$ [(2): Definition 2.1]. Any quasi-weight φ on $\mathscr{A}$ induces a closed $*$-representation π_φ of $\mathscr{A}$ on a Hilbert space $\mathcal{H}_\varphi$, which is called the GNS-representation of $\mathscr{A}$ for φ. A quasi-weight φ on $P(\mathcal{N}_\varphi)$ in $\mathscr{A}$ is said to be *admissible* if π_φ is bounded. A weight φ on $P(\mathscr{A})$ is said to be *admissible* if the quasi-weight $\varphi_q := \varphi\lceil_{P(\mathcal{N}_\varphi)}$ defined by φ is admissible, where $\mathcal{N}_\varphi := \{x \in \mathscr{A};\ \varphi(x^*a^*ax) < \infty \text{ for all } a \in \mathscr{A}\}$. For the admissibility of (quasi-) weights in locally convex $*$-algebras, we have the following

Theorem 2. [(2): Theorem 3.12]. *Let $\mathscr{A}$ be a locally convex $*$-algebra with $\mathscr{A} = \mathscr{A}_0$.*

(1) If $\mathscr{A}$ is pseudo-complete, then every quasi-weight and every weight is admissible, and $|a|_\infty := \sup_{\varphi\in W_q(\mathscr{A})} \|\pi_\varphi(a)\| \leqq \beta(a^*a)^{\frac{1}{2}}$ for all $a \in \mathscr{A}$, where $W_q(\mathscr{A})$ is the set of all quasi-weights in $\mathscr{A}$.

(2) If φ is a quasi-weight on $P(\mathcal{N}_\varphi)$ in $\mathscr{A}$ such that φ is continuous on the subspace $D(\varphi)$ of $\mathscr{A}$ generated by $\{x^*x;\ x \in \mathcal{N}_\varphi\}$, then φ is admissible.

Furthermore, we have defined the notion of *approximately admissible* (quasi-) weight in $*$-algebras in [(2): Definition 5.1] and characterized it in [(2): Theorem 5.2]. The results obtained have been applied to vector weights and tracial weights on unbounded operators, as well as to weights on smooth subalgebras of a C^*-algebra.

In (3) we have defined the notion of unbounded C^*-seminorms and constructed unbounded $*$-representations from them. Furthermore, we have defined and studied the notions of spectrality and stability of unbounded C^*-seminorms. Let $\mathscr{A}$ be a $*$-algebra. A C^*-seminorm p on a $*$-subalgebra $D(p)$ of $\mathscr{A}$ is said to be an *unbounded C^*-seminorm* on $\mathscr{A}$ with the domain $D(p)$. Let p be an unbounded C^*-seminorm on $\mathscr{A}$. Then $\ker p := \{x \in D(p);\ p(x) = 0\}$ is a $*$-ideal of $D(p)$ and $\mathcal{N}_p := \{x \in D(p);\ ax \in D(p), ^\forall \in \mathscr{A}\}$ is a left ideal of $\mathscr{A}$, and the quotient $*$-algebra $D(p)|\ker p$ is a normed $*$-algebra with the C^*-norm $\|x + \ker p\|_p := p(x)$, $x \in D(p)$. Let $\mathscr{A}_p$ be the C^*-algebra obtained by the completion of $D(p)|\ker p$, and $\mathrm{Rep}(\mathscr{A}_p)$ the set of all faithful nondegenerate $*$-representations Π_p of the C^*-algebra $\mathscr{A}_p$ on Hilbert spaces $\mathcal{H}_{\Pi_p}$. For any $\Pi_p \in \mathrm{Rep}(\mathscr{A}_p)$ we put $D(\pi_p) :=$ linear span of $\{\Pi_p(x + \ker p)\xi; x \in \mathcal{N}_p,\ \xi \in \mathcal{H}_{\Pi_p}\}$, $\pi_p(a)(\sum_k \Pi_p(x_k + \ker p)\xi_k) = \sum_k \Pi_p(ax_k + \ker p)\xi_k$ (finite sums) for $a \in \mathscr{A}$, $\{x_k\} \subset \mathcal{N}_p$ and $\{\xi_k\} \subset \mathcal{H}_{\Pi_p}$. Then we have the following

Proposition 3. [(3): Proposition 2.2]. *Let p be an unbounded C^*-seminorm on a $*$-algebra $\mathscr{A}$. For any $\Pi_p \in Rep(\mathscr{A}_p)$ there exists a $*$-representation π_p of $\mathscr{A}$ on a Hilbert space $\mathcal{H}_{\Pi_p}$ such that $\|\overline{\pi_p(b)}\| \leqq p(b)$ for each $b \in D(p)$ and $\|\overline{\pi_p(x)}\| = p(x)$ for each $x \in \mathcal{N}_p$.*
Here we put

$$\begin{aligned} \mathrm{Rep}(\mathscr{A}, p) &:= \{\pi_p;\ \Pi_p \in \mathrm{Rep}(\mathscr{A}_p)\}, \\ \mathrm{Rep}^{WB}(\mathscr{A}, p) &:= \{\pi_p \in \mathrm{Rep}(\mathscr{A}, p);\ \mathcal{H}_{\pi_p} = \mathcal{H}_{\Pi_p}\}. \end{aligned}$$

An unbounded C^*-seminorm p on $\mathscr{A}$ is said to be *semifinite* if $\mathcal{N}_p$ is dense in $D(p)$ with respect to the seminorm p, and it is said to be *weakly semifinite* if $\mathrm{Rep}^{WB}(\mathscr{A}, p) \neq \emptyset$. An element π_p of $\mathrm{Rep}^{WB}(\mathscr{A}, p)$ is said to be a *well-behaved $*$-representation* in $\mathrm{Rep}(\mathscr{A}, p)$. By Proposition 3 we constructed a family $\mathrm{Rep}(\mathscr{A}, p)$ of strongly nondegenerate $*$-representations of $\mathscr{A}$ from an unbounded C^*-seminorm p on $\mathscr{A}$. Here we say that a $*$-representation π of $\mathscr{A}$ is *strongly nondegenerate* if there exists a left ideal $\mathcal{I}$ of $\mathscr{A}$ contained in the bounded part $\mathscr{A}_b^\pi := \{x \in \mathscr{A};\ \overline{\pi(x)} \in B(\mathcal{H}_\pi)\}$ of π such that $\overline{\pi(\mathcal{I})}\mathcal{H}_\pi$ is total in $\mathcal{H}_\pi$. Conversely we can define an unbounded C^*-seminorm r_π for a strongly nondegenerate $*$-representation π of $\mathscr{A}$ by

$$\begin{cases} D(r_\pi) = \mathscr{A}_b^\pi := \{x \in \mathscr{A};\ \pi_b(x) := \overline{\pi(x)} \in B(\mathcal{H}_\pi)\} \\ r_\pi(x) = |\pi_b(x)\|,\ x \in D(r_\pi). \end{cases}$$

Here we can define a faithful $*$-representation $\Pi_{r_\pi}^N$ of the C^*-algebra $\mathscr{A}_{r_\pi}$ on the Hilbert space $\mathcal{H}_\pi$ by $\Pi_{r_\pi}^N(x + N_{r_\pi}) := \pi_b(x)$, $x \in D(r_\pi)$. The $*$-representation $\pi_{r_\pi}^N$ of $\mathscr{A}$ defined by $\Pi_{r_\pi}^N$ is said to be the *natural representation of $\mathscr{A}$ induced by π* and we have obtained some results for it: [(3): Proposition 3.1, 3.4 and 3.6].
In (1) we defined the notions of (hereditary) spectrality and stability of C^*-seminorms and of bounded $*$-representations. Here we have defined such notions of unbounded C^*-seminorms and of unbounded $*$-representations, and obtained the following results:

Theorem 4. [(3): Theorem 6.8]. *The following statements are equivalent:*

(i) There exists a strongly nondegenerate $*$-representation π of $\mathscr{A}$ such that π_b is (hereditary) spectral.

(ii) There exists a maximal, weakly semifinite, (hereditary) spectral unbounded C^*-seminorm on $\mathscr{A}$.

Theorem 5. [(3): Theorem 6.10] *Let $\mathscr{A}$ be a $*$-algebra and p a semifinite unbounded C^*-seminorm on $\mathscr{A}$. Then the following statements are equivalent:*

(i) p is hereditary spectral.

(ii) p is spectral and stable.

Our paper (4) has dealt with the spectral invariance of locally convex $*$-algebras, the K-theory of isomorphism at a general level and the differential structure of C^*-algebras as an application. Let $\mathscr{A}$ be a locally convex $*$-algebra and $C\mathrm{Rep}(\mathscr{A})$ the family of all continuous bounded $*$-representations of $\mathscr{A}$. If $C\mathrm{Rep}(\mathscr{A}) \neq \emptyset$, then $\mathscr{A}$ is *representable*. Then we define a C^*-seminorm on $\mathscr{A}$ by

$$|x|_u := \sup\{\|\pi(x)\|;\ \pi \in C\mathrm{Rep}(\mathscr{A})\},\ x \in \mathscr{A}$$

and it is said to be the *Gelfand-Naimark C^*-seminorm* on $\mathscr{A}$. The C^*-algebra $\mathscr{A}_{|\cdot|_u}$ constructed from the C^*-seminorm $|\cdot|_u$ is said to be an *enveloping C^*-algebra* of $\mathscr{A}$ and denoted by $E(\mathscr{A})$. The natural map $j:\ x \in \mathscr{A} \to x + N_{|\cdot|_u} \in E(\mathscr{A})$ is a $*$-homorphism. If $\mathscr{A}$ is representable and $S_{P_{\mathscr{A}}}(x) = S_{P_{E(A)}}(j(x))$ for each $x \in \mathscr{A}$, then $\mathscr{A}$ is said to be *spectral invariance*, and if for $x \in \mathscr{A}$ and a function f holomorphic on $S_{PE(A)}(j(x))$ there exists $y \in \mathscr{A}$ such that $f(j(x)) = j(y)$, then $\mathscr{A}$ is said to be *local.* The spectral invariance of $\mathscr{A}$ can be characterized by the (C^*-)spectarality and the locality of $\mathscr{A}$ as follows:

Theorem 6. [(4): Theorem 2.11]. *The following statements are equivalent:*

(i) $\mathscr{A}$ is spectral invariant.

(ii) $\mathscr{A}$ is C^*-spectral.

(iii) $\mathscr{A}$ is spectral and hermitian.

(iv) $\mathscr{A}$ is local and rad $\mathscr{A}$ = srad $\mathscr{A}$.

Furthermore, the spectral invariance of $\mathscr{A}$ can be characterized by the stability of $\mathscr{A}$ and the existence of spectral continuous bounded $*$-representations of $\mathscr{A}$ in [(4): Theorem 2.15]. We have applied Theorem 9 to the K-theory isomorphism of Fréchet locally m-convex $*$-algebras [(4): Theorem 3.1] and investigated the properties of C^*-spectrality and spectral invariance of the Fréchet $*$-algebra defined by a differential seminorm as a typical application of these results.

Our paper (7) has dealt with a general approach to the differential structure of C^*-algebras proposed by Blackdar and Cuntz [BC]. The smoothness properties of differential Fréchet algebras defined by (not necessarily l^1-summable) differential norms have been investigated. They have been used, by taking appropriate projective limits and inductive limits, to construct and investigate classes of non-commutative smooth algebras describing differential structures defined by differential norms. Let $\mathscr{A}$ be a dense $*$-subalgebra of a C^*-algebra $\tilde{\mathscr{A}}$, and ω^+ the set of all nonnegative sequences in $\mathbb{R}_+$. A *differential seminorm* on $\mathscr{A}$ is a mapping $T:\ \mathscr{A} \to \omega^+$, $x \to T(x) = (T_0(x), T_1(x), T_2(x), \cdots)$ satisfying the following

(i): $T_0(x) \leqq C\|x\|$ for all $x \in \mathscr{A}$,

(ii): $T(x+y) \leqq T(x) + T(y)$, $T(\lambda x) = |\lambda| T(x)$ for all $x, y \in \mathscr{A}$ and $\lambda \in \mathbb{C}$,

(iii): $T(xy) \leqq T(x)T(y)$ (convolution product).

We additionally assume that

(iv): $T_k(x^*) = T_k(x)$ for $^\forall x \in \mathscr{A}$ and $^\forall k \in \mathbb{N}_0$.

B-C further assumed that $(T_k(x))$ is l^1-summable for each $x \in \mathfrak{A}$. We don't require T to be l^1-summable. We can assume $T_0(x) = \|x\|$ making T a norm. The powers of a derivation naturally defines a differential seminorm by defining $T_k(x) := \|\delta^k(x)\|/k!$. A differential seminorm T is *closable* if any T-Cauchy sequence converges provided it converges in norm. B-C have shown that a differential seminorm is determined by a linear map with nonnegative norm-curvature into a graded Banach $*$-algebra. The added generality of not demanding l^1-summability has led to the following: For $k \in \mathbb{N}_0$, $P_k(x) := \sum_{i=0}^k T_i(x)$ defines a sequence of submultiplicative $*$-seminorms, and the completion of $\mathscr{A}$ in P_k is a Banach $*$-algebra $\mathscr{A}_{(k)}$ (or $C^k(\mathscr{A}, T)$). Then $\mathscr{A}_\tau$ (or $C^\infty(\mathscr{A}, T)) :=$ Proj lim $\mathscr{A}_{(k)}$ is a Fréchet $*$-algebra whose bounded part $\mathscr{A}_T$ is a Banach $*$-algebra. These are contained in $\tilde{\mathscr{A}}$ since T is assumed closable. The inclusion induces continuous homomorphisms φ and φ_k from $\mathscr{A}_\tau$ to $\tilde{\mathscr{A}}$, and from $\mathscr{A}_{(k)}$ to $\tilde{\mathscr{A}}$, respectively. Then we have the following

Theorem 7. [(7): Theorem 3.3]. *The following statements hold:*

(i) $\mathscr{A}_\tau$ is a C^*-spectral algebra with spectral C^*-seminorm $\|\cdot\|_\varphi := \|\varphi(\cdot)\|$.

(ii) $\mathscr{A}_\tau$ is a hermitian Q-algebra.

(iii) $\mathscr{A}_\tau$ is spectrally invariant in $\tilde{\mathscr{A}}$ via the map φ and $E(\mathscr{A}_\tau) = \tilde{\mathscr{A}}$.

(iv) $\mathscr{A}_\tau$ is local, that is, it is closed under homorphic functional calculus of $\tilde{\mathscr{A}}$ in the sense that given $x \in \mathscr{A}_\tau$ and a holomorphic function f on $S_{P_{\mathscr{A}}}(\varphi(x))$, there exists an element y of $\mathscr{A}_\tau$ such that $f(\varphi(x)) = \varphi(y)$.

(v) $\mathscr{A}_\tau$ and $\tilde{\mathscr{A}}$ have the same K-theory, that is, $K_*(\mathscr{A}_\tau) = K_*(\tilde{\mathscr{A}})$.

Furthermore, the same results hold for $\mathscr{A}_{(k)}$.

(vi) [(7): Theorem 3.4]: $\mathscr{A}_\tau$ is closed under the C^∞-functional calculus of $\tilde{\mathscr{A}}$ via φ in the sense that given $x = x^* \in \mathscr{A}_\tau$ and a C^∞-function f on $S_{P_{\tilde{\mathscr{A}}}}(\varphi(x))$, there exists an element y of $\mathscr{A}_\tau$ such that $f(\varphi(x)) = \varphi(y)$. The Banach $*$-algebra $\mathscr{A}_{(k)}$ is also closed under the C^∞-functional calculus of $\tilde{\mathscr{A}}$ via φ_k.

The algebras $\mathscr{A}_{(k)} := C^k(\mathscr{A}, T)$ and $\mathscr{A}_\tau := C^\infty(\mathscr{A}, T)$ represent the *C^k-structure defined by* T. We propose the following as the analytic structure defined by a differential norm T. An element x of $\mathscr{A}$ is said to be *T-analytic* if $p^t(x) := \sum_{k=0}^\infty t^k T_k(x) < \infty$ for some $t > 0$, and it is said to be *T-entire analytic* if $p^t(x) < \infty$ for all $t > 0$. We denote by $\mathscr{A}^\omega$ (resp. $\mathscr{A}^{e\omega}$) the set of all T-analytic (resp. T-entire analytic) elements of $\mathscr{A}$, namely

$$\mathscr{A}^\omega = \cup\{\mathscr{A}^\omega(t);\ t > 0\} \quad \text{and} \quad \mathscr{A}^{e\omega} = \cap\{\mathscr{A}^\omega(t);\ t > 0\}.$$

If $\mathscr{A} = \mathscr{A}^\omega$ (resp. $\mathscr{A} = \mathscr{A}^{e\omega}$), Then T is said to be *analytic* (resp. *entire analytic*). Let $\mathscr{A}^\omega(t)^\sim$ be the completion of the normed $*$-algebra $\mathscr{A}^\omega(t)[p^t]$ and $(\mathscr{A}^\omega)^\sim =$ ind $\lim_{t\to 0} \mathscr{A}^\omega(t)^\sim$ be complete locally convex $*$-algebra and $(\mathscr{A}^{e\omega})^\sim =$ Proj $\lim_{t\to\infty} \mathscr{A}^\omega(t)^\sim$ be Fréchet $*$-algebra. Then we have the following

Theorem 8. [(7): Theorem 3.7]. *Let T be a differential norm on $\mathscr{A}$. Then the following statements hold:*

(i) Suppose that T is analytic on $\mathscr{A}$. Then $(\mathscr{A}^{\omega})^{\sim}$ is C^*-spectral with spectral C^*-seminorm $\|\cdot\|_{\varphi}$, is an hermitian Q-algebra and local.

(ii) Suppose T is entire analytic on $\mathscr{A}$. Then $(\mathscr{A}^{e\omega})^{\sim}$ is C^*-spectral with spectral C^*-seminorm $\|\cdot\|_{\varphi}$, and is an hermitian Q-algebra. Furthermore, if T is closable, then $(\mathscr{A}^{e\omega})^{\sim}$ is local.

Similarly, we can define

$$\mathscr{A}_{\tau}^{\omega} = \cup\{\mathscr{A}_{\tau}^{\omega}(t);\ t>0\} = \operatorname{ind}\lim_{t\to 0}\mathscr{A}_{\tau}^{\omega}(t),$$
$$\mathscr{A}_{\tau}^{e\omega} = \cap\{\mathscr{A}_{\tau}^{\omega}(t);\ t>0\} = \operatorname{Proj}\lim_{t\to\infty}\mathscr{A}_{\tau}^{\omega}(t),$$

where $\mathscr{A}_{\tau}^{\omega}(t) := \{x \in \mathscr{A}_{\tau};\ p^t(x) < \infty\}$. We have the same results for $\mathfrak{A}_{\tau}^{\omega}$ and $\mathfrak{A}_{\tau}^{e\omega}$ as those of Theorem 8 [(7): Theorem 3.10]. We define by Λcd the set of all closable derived norms ((7): Definition 4.2) on $\mathscr{A}$ and denote by $\tau(\Lambda_{cd})$ the locally convex topology on $\mathscr{A}$ defined by Λ_{cd}. The completion $\tilde{\mathscr{A}}[\tau(\Lambda_{cd})]$ of $\mathscr{A}$ is said to be *smooth envelope* of $\mathscr{A}$ denoted by $S\mathscr{A}$. The algebra $\mathscr{A}$ is said to be *smooth* if $S\mathscr{A} = \mathscr{A}$, that is, $\mathscr{A}$ is complete in the topology defined by all closable derived norms on $\mathscr{A}$. For a smooth algebra in a C^*-algebra we have the following

Theorem 9. [(7): Theorem 4.8] *Let $\mathscr{A}$ be a smooth algebra in a C^*-algebra $\tilde{\mathscr{A}}$. Then the following statements hold:*

(i) $\mathscr{A}$ is a dense $*$-subalgebra of $\tilde{\mathscr{A}}$ continuously embedded in $\tilde{\mathscr{A}}$.

(ii) $\mathscr{A}$ is a complete locally m-convex $*$-algebra.

(iii) $\mathscr{A}$ is a hermitian Q-algebra.

(iv) $\mathscr{A}$ is spectrally invariant and $E(\mathscr{A}) = \tilde{\mathscr{A}}$.

(v) $\mathscr{A}$ is closed under holomorphic functional calculus of $\tilde{\mathscr{A}}$, and $\tilde{\mathscr{A}}$&$\mathscr{A}$ have same K-theory.

(vi) $\mathscr{A}$ is closed under the C^{∞}-functional calculus of self-adjoint elements.

Furthermore, we have developed an approach to the C^k-structures based on the derived norms of total order $\leqq k$ which is along the lines of the approach to smooth algebras. This gives rise to C^{∞}-envelope $C^{\infty}\mathscr{A}$. If $\mathscr{A} = C^{\infty}(\mathscr{A})$, then $\mathscr{A}$ is called a C^{∞}-algebra. For C^{∞}-algebras we have obtained similar results to those of Theorem 9 [(7): Theorem 5.12].

Our paper (8) has treated with the (spectral) well-behavedness of unbounded $*$-representations of *locally convex $*$-algebras* and the existence of (spectral) well-behaved $*$-representations of *locally convex $*$-algebras* by unbounded C^*-seminorms. In (3) we have constructed unbounded $*$-representations of $*$-*algebra* on the basis of unbounded C^*-seminorms and investigated a class of well-behaved $*$-representations. K. Schmüdgen [Sc, 2] has defined another (but related) notion of well-behaved $*$-representations. Those notions were considered in order to avoid pathologies which may appear for general $*$-representations and to select "nice" representations which may have a rich theory. In order to consider the case of locally convex $*$-algebras instead of $*$-algebras we have defined the notions of *uniformly nondegenerate* $*$-representations and *topologically semifinite* unbounded C^*-seminorms in Definition 3.2 and 3.3 in (8), and obtained the following result:

Theorem 10. [(8): Theorem 3.4]. *Let $\mathscr{A}$ be a pseudo-complete locally convex $*$-algebra with identity. Then the following statements are equivalent:*

(i) There exists a well-behaved $*$-representation of $\mathscr{A}$, that is, there exists a topologically semifinite unbounded C^*-seminorm on $\mathscr{A}$.

(ii) There exists a uniformly nondegenerate $*$-representation of $\mathscr{A}$.

(iii) There exists an unbounded C^*-seminorm p on $\mathscr{A}$ satisfying the representability condition:

$$\text{(UR)}: \quad \mathcal{N}_p \cap \mathcal{I}_b \not\subset N_p,$$

where $\mathcal{I} := \{x \in \mathscr{A}_b;\ ax \in \mathscr{A}_b,\ \forall a \in \mathscr{A}\}$.

To characterize the existence of spectral well-behaved $*$-representations of locally convex $*$-algebras $\mathscr{A}$, we have defined the notions of *(hereditary) spectral $*$-representations* of $\mathscr{A}$, *(hereditary) spectral unbounded C^*-seminorms* on $\mathscr{A}$ and *spectral invariance* of $\mathscr{A}$ in Definition 4.1, 4.2 and 4.7 in (8), respectively, and showed that

Theorem 11. [(8): Theorem 4.8]. *Let $\mathscr{A}$ be a pseudo-complete locally convex $*$-algebra with identity. The following statements are equivalent:*

(i) There exists a spectral tw-semifinite unbounded C^*-seminorm on $\mathscr{A}$ whose domain contains the bounded part $\mathscr{A}_b$ of $\mathscr{A}$.

(ii) There exists a spectral well-behaved $*$-representation of $\mathscr{A}$.

(iii) There exists a spectral uniformly nondegenerate $*$-representation of $\mathscr{A}$.

(iv) $\mathscr{A}$ is spectral invariant.

Furthermore, we have considered the existence of hereditary spectral well-behaved $*$-representations of locally convex $*$-algebras and the diration-problem in case of locally convex $*$-algebras, and obtained the results: Proposition 4.12 and Theorem 5.2 in (8).

Our paper (9) has dealt with the algebra $\Omega_\infty\mathscr{A}$ and $\Omega_\varepsilon\mathscr{A}$ obtained by taking respectively the projective limit and the inductive limit Banach $*$-algebras obtained by completing the universal graded differential algebra $\Omega^*\mathscr{A}$ of abstract non-commutative differential forms over a C^*-normed algebra A which is either a Banach $*$-algebra or a Fréchet $*$-algebra. Various quantized integrals on $\Omega_\infty\mathscr{A}$ induced by a K-cocycle on A are considered. The GNS-representation of $\Omega_\infty A$ defined by a d-dimensional non-commutative volume integral on a d^+-summable K-cocycle on A is realized as the representation induced by the left action of A on $\Omega^* A$. This supplements the representation A on the space of forms discussed by Connes [Co.VI. 1, Prop. 5, p.550]. Let $\mathscr{A}[\|\cdot\|]$ be a C^*-normed algebra with identity and $\tilde{\mathscr{A}} := \tilde{\mathscr{A}}[\|\cdot\|]$ be the C^*-algebra completion of $\mathscr{A}$. Let $\Omega^*\mathscr{A}$ be the universal graded differential algebra over $\mathscr{A}$ [Co, Ch.III.1, p.185]. We consider the limit algebras of $\Omega^*\mathscr{A}$ in the following situations:

(i): $\mathscr{A}$ is a Banach $*$-algebra with a norm $|\cdot|$.

(ii): $\mathscr{A}$ is a Fréchet $*$-algebra with a topology defined by a sequence of seminorms $\{|\cdot|_n\}$.

These are prototype situations that occur frequently.

When $\mathscr{A}$ is a Banach $*$-algebra $\mathscr{A}[|\cdot|]$: The complete norm $|\cdot|$ on $\mathscr{A}$ is necessarily finer than the C^*-norm $\|\cdot\|$. Following $[A_r]$ and [Co, p.373], the following system of norms is defined on $\Omega^*\mathscr{A}$

$$\left|\omega := \sum_{k=0}^{\infty} \omega_k\right| = \sum_{k=0}^{\infty} r^k |\omega_k|_\pi, \quad r \in \mathbb{R}^+,$$

where $\omega_k \in \Omega^k\mathscr{A}$ is the k^{th} degree part of ω, and $|\cdot|_\pi$ is the projective tensor product norm on the space $\Omega^k\mathscr{A} \simeq \mathscr{A} \otimes \bar{\mathscr{A}}^{\otimes k}$ of forms of degree k arising from the complete norm $|\cdot|$ on $\mathscr{A}$. Let

$$\begin{aligned} \Omega_r\mathscr{A} &:= \widetilde{\Omega^*\mathscr{A}}[|\cdot|_r] \quad \text{the completion} \\ &= \{\omega = \sum_{k=0}^{\infty} \omega_k;\ \omega_k \in \Omega^k\mathscr{A},\ ^\forall k \text{ and } \sum_{k=0}^{\infty} r^k |\omega_k|_\pi < \infty\} \end{aligned}$$

be a Banach $*$-algebra with norm $|\omega|_r := \sum_{k=0}^{\infty} r^k |\omega_k|_\pi$. The following two limit algebras are formed with this system of Banach $*$-algebras.

(a): (Arveson) $\Omega_\infty\mathscr{A} :=$ Proj $\lim_{\to\infty} \Omega_r\mathscr{A}$.

(b): (Connes) $\Omega_\varepsilon\mathscr{A} :=$ ind $lim_{r\to 0}\Omega_r\mathscr{A}$.

Let $\varepsilon_r : \Omega_r\mathscr{A} \mapsto \mathscr{A}$, $\varepsilon_r(\sum_0^\infty \omega_k) := \omega_0$. It is a surjective $*$-homomorphism. Then we have the following

Proposition 12. [(9): Proposition 2.1].

(1) The algebra $\Omega_\infty\mathscr{A}$ is a Fréchet $*$-algebra whose bounded part $(\Omega_\infty\mathscr{A})_b$ is $\mathscr{A}$.

(2) There exist continuous $*$-homomorphisms $\varphi_r : C^*(\Omega_r\mathscr{A}) \mapsto \tilde{\mathscr{A}}$, $\varphi : E(\Omega_\infty\mathscr{A}) \mapsto \tilde{\mathscr{A}}$ where $C^*(\Omega_r\mathscr{A})$ (resp. $E(\Omega_\infty\mathscr{A})$) is the enveloping C^*-algebra of $\Omega_r\mathscr{A}$ (resp. the enveloping σ-C^*-algebra of $\Omega_\infty\mathscr{A}$).

For the algebra $\Omega_\varepsilon\mathscr{A}$ we have the following

Theorem 13. [(9): Theorem 2.2].

(1) $\Omega_\varepsilon\mathscr{A}$ is a locally m-convex m-barrelled Q-algebra and $\Omega^*\mathscr{A}$ is sequentially dense in $\Omega_\varepsilon\mathscr{A}$.

(2) srad Omega$_\varepsilon\mathscr{A}$ = rad $\Omega_\varepsilon\mathscr{A} = \ker\varepsilon$, where ε is a continuous surjective $*$-homomorphism of $\Omega_\varepsilon\mathscr{A}$ onto $\mathscr{A}$ defined by $\varepsilon(\sum_0^\infty \omega_k) = \omega_0$.

(3) $\Omega_\varepsilon\mathscr{A}$ is a spectral algebra, and its enveloping pro-C^*-algebra $E(\Omega_\varepsilon\mathscr{A})$ is isomorphic to the enveloping C^*-algebra $C^*(\mathscr{A})$ of $\mathscr{A}$.

(4) If $\mathscr{A}$ is spectral invariance in $\tilde{\mathscr{A}}$, then $\Omega_\varepsilon\mathscr{A}$ is a C^*-spectral algebra satisfying $E(\Omega_\varepsilon\mathscr{A}) = \tilde{\mathscr{A}}$, and $\Omega_\varepsilon\mathscr{A}$ and $\tilde{\mathscr{A}}$ have the same K-theory.

When $\mathscr{A}$ is a Fréchet $*$-algebra $\mathscr{A}[\{|\cdot|_n\}]$: In this case we assume that the enveloping σ-C^*-algebra $E(\mathscr{A})$ of $\mathscr{A}$ is the C^*-algebra $\tilde{\mathscr{A}}$ and that each $|\cdot|_n$ is closable with respect to the C^*-norm $\|\cdot\|$. Since $E(\mathscr{A}) = \tilde{\mathscr{A}}$, the C^*-norm $\|\cdot\|$ on $\mathscr{A}$ is the greatest C^*-seminorm on $\mathscr{A}$, automatically continuous in the topology t defined by the sequence $\{|\cdots|_n\}$ of seminorms assumed increasing without loss of generality. Thus we can assume that $\|\cdot\| \leqq |\cdot|_n$ for all $n \in \mathbb{N}$ and $\{|\cdot|_n\}$ is increasing. Here $\mathscr{A}_n := \tilde{\mathscr{A}}[|\cdot|_n]$ completion of $\mathscr{A}$ in $|\cdot|_n$. The closability of $|\cdot|_n$ with respect to $\|\cdot\|$ implies that $\mathscr{A}_n \subset \tilde{\mathscr{A}}$ and by the closability of $|\cdot|_n$ with respect to $|\cdot|_m$ for any $n > m$. $\mathscr{A}_n \subset \mathscr{A}_m$ for $n > m$. Hence $\mathscr{A} = \text{Proj lim}\,\mathscr{A}_n = \cap_{n=1}^\infty \mathscr{A}_n$. This makes available the techniques and results above. For each $r > 0$ and $n \in \mathbb{N}$, let $|\cdot|_{n,r}$ be the norm on $\Omega^*\mathscr{A}$ defined by $|\omega = \sum_{n=0}^\infty \omega_k|_{n,r} := \sum_{k=0}^\infty r^k|\omega_k|_{n,\pi}$, where $|\cdot|_{n,\pi}$ is the projective cross-norm on $\Omega^k\mathscr{A}$ arising from $|\cdot|_n$. Then, $(\Omega^*\mathscr{A})^\sim_{n,r} = \widetilde{\Omega^*\mathscr{A}}[|\cdot|_{n,r}]$ completion coincides with $\Omega_r\mathscr{A}_n = \widetilde{\Omega^*\mathscr{A}_n}[|\cdot|_{n,r}]$ completion. Let the Fréchet $*$-algebra $\Omega_r\mathscr{A}$ be the completion of $\Omega^*\mathscr{A}$ in the topology τ_r defined by the sequence of norms $\{|\cdot|_{n,r};\ n \in \mathbb{N}\}$. Then,

$$\Omega_r\mathscr{A} = \text{Proj}\lim_{n\to\infty}(\Omega^*\mathscr{A})^\sim_{n,r} = \text{Proj}\lim_{n\to\infty}\Omega_r\mathscr{A}_n = \cap_{n=1}^\infty \Omega_r\mathscr{A}_n.$$

Thus we have the following

$$\begin{aligned}
\Omega_\infty\mathscr{A} &= \text{Proj}\lim_{r\to\infty}\Omega_r\mathscr{A} = \text{Proj}\lim_{r\to\infty}\ \text{Proj}\lim_{n\to\infty}\Omega_r\mathscr{A}_n = \cap_{r=1}^\infty \cap_{n=1}^\infty \Omega_r\mathscr{A}_n,\\
\Omega_\varepsilon\mathscr{A} &= \text{ind}\lim_{r\to 0}\Omega_r\mathscr{A} = \text{ind}\lim_{r\to 0}\Big(\text{Proj}\lim_{n\to\infty}\Omega_r\mathscr{A}_n\Big) = \cup_{r=0}^\infty \cap_{n=1}^\infty \Omega_r\mathscr{A}_n.
\end{aligned}$$

We have investigated these limit algebras; in particular looked for the analogous of the results in Theorem 13.

Theorem 14. [(9): Theorem 4.3]. *Under the assumptions stated above, the following hold:*

(1) $\Omega_\varepsilon\mathscr{A}$ is a locally convex Q-algebra, and $\ker\varepsilon$ is a quasinilpotent ideal of $\Omega_\varepsilon\mathscr{A}$.

(2) The enveloping pro-C^*-algebra of $\Omega_\varepsilon\mathscr{A}$ is the C^*-algebra $\tilde{\mathscr{A}}$.

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