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A CONTRIBUTION TO UNIFORM AND TOPOLOGICAL ALGEBRAS

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1. Tribute

It was 1984, when I became a part of the Department of Mathematics, Sardar Patel University as an M. Sc. student, Professor Bhatt was adopting (and adapting to), for some time, the teaching of so called Applied Mathematics. Consequently, he taught me two courses on Differential Equations. It was third semester that brought us closer and we mutually agreed on my career with him for M. Phil. and Ph. D. In the fourth semester, I lost my brother in law of the age around 36. Professor Bhatt ran down to my home in Nadiad convincing my parents to let me live in hostel and he would arrange for my meal at his home. All this, and above all his reputation as a spiritual person, made my decision firm to work under his supervision. In my M. Phil. he planned my dissertation based on the first half of [Sch78]. It was during this time, that he started carving me for commutative Banach Algebras. He asked me to read *independently* the celebrated book by Ronald Larsen [Lar73]. He would drive me out of his office if I sought his help on that book. While I was typing my dissertation, he gave me a research problem. On solving the same, he insisted to communicate the same as a single author. I was amazed at his integrity. Incidentally, the paper was rejected as the result was already there in the latter sections of [Sch78]!

In 1987, I enrolled for Ph. D. under joint supervision of Professor Bhatt and Professor Vasavada. In 1988, we obtained the results of [3] but after rejections from a number of journals, it was published in 1992. Meanwhile in September 1989, he changed my topic and assigned to read [Phi88]. This is how I landed in noncommutative topological algebras, with unbounded objects.

After he met with an accident, when I paid a visit to him in the hospital, he had just come out of General Anesthesia and still had nausea because of after effects of GA. His wit was on the peak even at that time. He cut the smallest joke *Theorem J* and I responded with a small joke *Wikipedia.* Both of us burst into laughter. I did not know that the smallest jokes would become the last exchange between us.

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Professor Bhatt has influenced my life in this birth of mine to a great great extent. I do not know when the *Kaal Chakra* will give me an opportunity to repay my *Guruji*! It is because of this very thought of dying with the debt without repayment that I am still under the trauma of his sad demise.

2. Our Joint Research

For brevity, we shall define only those terms which, we feel, have not been so common. A will be an algebra over $\mathbb{R}$ or $\mathbb{C}$. A norm $\|\cdot\|$ on A is an *algebra norm* if it satisfies $\|xy\| \le \|x\|\|y\|$ for all $x, y \in A$. Also, $\|\cdot\|$ is a *norm with square property* if it satisfies $\|x^2\| = \|x\|^2$ for all $x \in A$. A *uniform norm* is an algebra norm with the square property. A Banach algebra $(A, \|\cdot\|)$ such that $\|\cdot\|$ is a uniform norm is called a *uniform Banach algebra.*

Theorem 2.1. *([3, Theorem 1])*

(i) Any linear norm with square property on a commutative algebra is an algebra norm.

(ii) Let $(A, \|\cdot\|)$ be a uniform Banach algebra. Let $|\cdot|$ be a linear norm with square property on A such that the set A^{-1} of all invertible elements forms an open set in $(A, |\cdot|)$. Then $\|\cdot\| = |\cdot|$.

Corollary 2.2. *([3]) Let $(A, \|\cdot\|)$ be a regular uniform Banach algebra. Let $|\cdot|$ be any norm on A such that $(A, |\cdot|)$ is a normed algebra. Then $\|\cdot\| \le |\cdot|$. Additionally, if $|\cdot|$ is a uniform norm, then $\|\cdot\| = |\cdot|$.*

Theorem 2.3. *([3, Theorem 2]) Let $(A, \|\cdot\|)$ be a uniform topological algebra that is a Q-algebra. Then the topology of A is normable and A is a uniform Banach algebra.*

In [3] we also discussed the connection of Michael problem with the uniform norm. As is already mentioned, this was my first topic for Ph. D. but the publication was delayed. The next topic was the Projective limits of C^*-algebras.

A *topological algebra* (A, τ) is an algebra with the Hausdorff topology τ in which addition, scalar multiplication and the ring multiplication are jointly continuous. A *seminorm* on an algebra A is a function $p : A \to \mathbb{R}$ satisfying all the properties of a norm except possibly that "$p(x) = 0 \Rightarrow x = 0$". Now the term *algebra seminorm* should be clear. An *involution* on an algebra A is a function $x \in A \mapsto x^* \in A$ satisfying the following properties for all $x, y \in A$ and $\lambda \in \mathbb{C}$.

(1) $(x + y)^* = x^* + y^*$;

(2) $(\lambda x)^* = \overline{\lambda} x^*$; and

(3) $(xy)^* = y^* x^*$.

Such an A is called a *-*algebra*. A C^*-*seminorm* on a *-algebra A is an algebra seminorm p on A that satisfies, for all $x \in A$,

(1) $p(x^*x) = p(x)^2$

(2) $p(x^*) = p(x)$

It is known that (1) is strong enough to imply that p is an algebra seminorm satisfying (2). A family P of seminorms on A is said to be *separating* if $x \in A, p(x) = 0$ for all $p \in P \Rightarrow x = 0$.

Let (A, τ) be a topological algebra. A separating family P of algebra seminorms is said to be an *m-calibration on A* if P determines the topology A in the sense that a net $(x_i)_{i\in I}$ in A converges to $x \in A$ if $p(x_i - x) \to 0$ for every $p \in P$. The set of all m-calibrations on A is denoted by $\mathscr{P}$. An *lmc algebra* is a topological algebra A whose topology is generated by an m-calibration. A *topological* *-*algebra* is a topological algebra A which is also *-algebra in which * is continuous. An *lmc* *-*algebra* is a topological algebra whose topology is generated by an m-calibration P consisting of *-algebra seminorms. Such a calibration is called an m^*-*calibration*. The set of all m^*-calibrations is denoted by $\mathscr{P}^*$.

For an lmc algebra (A, τ), $\mathscr{B}(\tau)$ denotes the collection of all absolutely convex, closed, bounded, idempotent subsets of A containing the identity $\mathbf{1}$. Further, if A is an lmc *-algebra, then $\mathscr{B}^*(\tau)$ is the collection of $B \in \mathscr{B}(\tau)$, which are closed under involution. For a subset S of A, $A(S) = \{\lambda x : \lambda \in \mathbb{C}, x \in S\}$. Given a calibration P on A, $S_P = \{x \in A : |x|_P = \sup_{p\in P} p(x) < \infty\}$. For a $P \in \mathscr{P}$ on a complete lmc algebra A, $S_P \in \mathscr{B}(\tau)$ and $(A(B), \|\cdot\|_B)$ is a Banach algebra with the Minkowski functional $\|\cdot\|_B$ of B. Conversely, given $B \in \mathscr{B}(\tau)$, there is $P \in \mathscr{P}$ on A such that $B \subset S_P$. The analogue for a complete lmc *-algebra holds too.

For two seminorms p, q on an algebra, we say that $q \leq p$ if there is $K > 0$ such that $q(x) \leq Kp(x)$ for all $x \in A$. This defines an order on every $P \in \mathscr{P}$. For two seminorms p_1, p_2 on an algebra, $p(x) = \max\{p_1(x), p_2(x)\}$, $(x \in A)$, is an algebra seminorm. In fact, if both p_1, p_2 are *-seminorms (respectively, C^*-seminorms, continuous seminors), then so is p. Given a complete lmc algebra A and $P \in \mathscr{P}$, we shall assume, without loss of generality, that P is directed. This can be achieved by adding to P, maximums of all finitely many seminorms in P.

Let A be a complete lmc algebra and $P \in \mathscr{P}$. For $p \in P$, $N_p = \{x \in A : p(x) = 0\}$ is a closed ideal of A. We adopt the notation $x_p = x + N_p \in A/N_p$. A/N_p is a normed algebra with the norm $\|x_p\|_p = p(x)$, $(x_p \in A/N_p)$. Also, A_p will denote the Banach algebra completion of $(A/N_p, \|\cdot\|_p)$. Then $A = \varprojlim_{p\in P} A_p$, the inverse limit of Banach algebras. For seminorms $q \leq p$ in P, $\chi_{p,q} : A/N_p \to A/N_q$ defined by $\chi_{p,q}(x_p) = x_q$, $(x_p \in A/N_p)$, is a continuous homomorphism and its unique continuous extension is also denoted by $\chi_{p,q} : A_p \to A_q$. Also, for each $p \in P$, the projection $\pi_p : A \to A_p$ is defined by $\pi_p(x) = x_p$, $(x \in A)$ and it is continuous. If A is a complete lmc *-algebra and $P \in \mathscr{P}^*$, then N_p is a closed *-ideal, A_p is a Banach* algebra and

$\pi_p, \chi_{p,q}$, $(p, q \in P, q \leq p)$, are continuous *-homomorphisms. Let A be a complete lmc algebra and $P \in \mathscr{P}$. For $p \in P$, $x \in A$, we define $D_p = \{f \in A' : f(1) = 1, |f(a)| \leq p(a)\}$, $V(A, p, x) = \{f(x) : f \in D_p\}$ and $D_P = \cup_{p \in P} D_p$. Also, for $x \in A$, $V(A, P, x) = \{f(x) : f \in D_P\}$ is called the *numerical range* of x. Indeed, the numerical range depends upon the choice of the calibration. The set of all *P-hermitian elements of A* is the set $H(A, P) = \{x \in A : V(A, P, x) \subset \mathbb{R}\}$ and the set of all *P-bounded elements of A* is $B_P = \{x \in A : V(A, P, x) \text{ is bounded}\}$, which is a Banach algebra with $|x|_P = \sup_{p \in P} p(x)$, $(x \in B_P)$.

On a complete lmc *-algebra A, if there is $P \in \mathscr{P}^*$ such that each $p \in P$ is a C^*-seminorm, then A is called a *pro-C^*-algebra*. In this case, $S(A)$ will denote the family of all continuous C^*-seminorms on A. Naturally, $S(A)$ is directed. For a pro-C^*-algebra A and $p \in S(A)$, $(A/N_p, \|\cdot\|_p)$ is automatically complete and $\|\cdot\|_p$ is a C^*-algebra norm. That is, $(A/N_p, \|\cdot\|_p)$ is a C^*-algebra. A complete metrizable pro-C^*-algebra is called *σ-C^*-algebra*. Let A be a pro-C^*-algebra. The C^*-algebra $b(A) = \{a \in A : \|x\|_\infty = \sup_{p \in S(A)} p(x) < \infty\}$ with $\|\cdot\|_\infty$ is called the *bounded part of A*. $b(A)$ is dense in A. Chronologically the results of [4] were obtained first, and hence, we shall follow that order.

The following is a stronger version of a result by [GK73]. We further improve the same as an application of [4, Theorem 3.1].

Theorem 2.4. *Let A be a complete lmc algebra with $P \in \mathscr{P}$. Then*

(1) $B_P = A(S_P)$ and $|\cdot|_P = |\cdot|_{S_P}$. Further, for each $x \in B_P$, $\overline{V(A, P, x)} = V(B_P, |\cdot|_P, x)$.

(2) (Vidav and Palmer) $A = H(A, P) \oplus iH(A, P)$.

Theorem 2.5. *([4, Theorem 3.1]) Let A be a complete lmc *-algebra. Then A is a pro-C^*-algebra iff A contains a *-algebra B such that*

(i) B is a C^*-algebra with some norm $\|\cdot\|$; and

(ii) the inclusion map $(B, \|\cdot\|) \to A$ is a continuous embedding with dense range.

Further, if $\{x \in B : \|x\| \leq 1\}$ is closed in A, then $B = b(A)$.

As an application, we obtained some characterizations of a pro-C^*-algebra A in terms of

(1) a dense subalgebra on which there is some involution having certain properties. Here A is not assumed to be a *-algebra;

(2) existence of the largest element of $\mathscr{B}^*(\tau)$;

(3) hermitian elements;

(4) the relation between continuous hermitian linear functionals and continuous positive linear functionals on A.

We also proved that if a *-subalgebra E of a pro-C^*-algebra A is a Banach* algebra with any norm, then $E \subset b(A)$, and this embedding is automatically continuous. We now state one more characterization of a pro-C^*-algebra that is an analogue of[BD73a, Theorem 31.10] and also supplements the Theorem 2.4(2), that is, the Vidav-Palmer Theorem.

Theorem 2.6. *([4, Theorem 4.4]) For a complete lmc algebra A with a directed $P \in \mathscr{P}$, the following are equivalent.*

(1) There is an involution * on A making A a pro-C^*-algebra and $P \subset S(A)$.

(2) $A' = H((A,P)') \oplus iH((A,P)')$.

(3) $A' = H((A,P)') \cap iH((A,P)') = \{0\}$.

Then the paper discusses when a pro-C^*-algebra is a C^*-algebra or finite dimensional or nuclear as a locally convex space. We shall come across the nuclear pro-C^*-algebra as C^*-nuclearity later on. Finally the paper discusses the unbounded representations and Kadison's transitivity. We curtail the list of the results discussed in the paper in explicit statements as well as many of them in the running discussion without binding it in separate statements.

The next paper is [1] that discusses the completely positive maps, tensor products and as an application C^*-nuclearity. First it establishes the importance of the matrix structure on a pro-C^*-algebra in the investigation. For brevity, we shall restrict stating only some results as a separate statement, some will be omitted and some will be woven in the running discussion. Given a pro-C^*-algebra A, we introduced the study of $M_n(A)$, the pro-C^*-algebra, in a unique way, of all $n \times n$ matrices with entries from A. It turns out that $M_n(A) = A \otimes M_n(\mathbb{C})$. Also, $b(M_n(A)) = M_n(b(A))$. Let A, B be pro-C^*-algebras and $\phi : A \to B$ be a linear map. Then for every $n \in \mathbb{N}$ we define $\phi_n : M_n(A) \to M_n(B)$ by $\phi([a_{ij}]) = [\phi(a_{ij})]$, $([a_{ij}] \in M_n(A))$.

The following result is obtained as an analogue of Stinespring dilation theorem. Here H, K will denote Hilbert spaces. For $E \subset H$, $[E]$ will denote the closed linear span of E and for a subalgebra M of B(H), M' will denote the commutant of M in $B(H)$.

Theorem 2.7. *([1, Theorem 2.2]) Let A be a pro-C^*-algebra.*

(1) If $\pi : A \to B(K)$ is a continuous *-representation of A and $V \in B(H,K)$, then $\phi : A \to B(H)$ defined by $\phi(x) = V^*\pi(x)V$, $(x \in A)$, is a continuous completely positive map.

(2) If $\phi : A \to B(H)$ is a continuous completely positive map, then there exist a Hilbert space K, a continuous *-representation $\pi : A \to B(K)$, a normal representation $\rho : \phi(A)' \to B(K)$ and $V \in B(H,K)$ such that $\phi(a) = V^*\pi(a)V$, $(a \in A)$, $\rho(x)V = Vx$, $(x \in \phi(A)')$, $\rho(\phi(A)') \subset \pi(A)'$ and $K = [\pi(A)VH]$.

As a corollary we obtain the Arveson's inequality for completely positive maps. A completely positive map between two pro-C^*-algebras preserve the bounded elements. It is also shown that two pro-C^*-algebras A, B are homeomorphically *-isomorphic iff there is a unital bijective continuous completely positive map with continuous completely positive inverse. Further, if

A, B are σ-C^*-algebras, then they are homeomorphically *-isomorphic iff they are matricially order isomorphic. It is also proved that a positive linear map ϕ between two pro-C^*-algebras maps bounded sets to a bounded sets and if the pro-C^*-algebras are σ-C^*-algebras, then ϕ is automatically continuous.

We then turn to the investigation of tensor products of two pro-C^*-algebras. On the algebraic tensor product $A \otimes B$, we define the

(1) **projective tensorial topology** π**,** making $A \otimes B$ an lmc *-algebra.
(2) **topology** ε **of biequicontinuous convergence** making $A \otimes B$ a locally convex space.
(3) **projective tensorial pro-**C^***-topology** γ, making $A \otimes B$ a pro-C^*-algebra.
(4) **injective tensorial pro-**C^***-topology** α, making $A \otimes B$ a pro-C^*-algebra.

The completions of the above will be denoted by $A \hat{\underset{\pi}{\otimes}} B$, $A \hat{\underset{\varepsilon}{\otimes}} B$, $A \hat{\underset{\gamma}{\otimes}} B$, $A \hat{\underset{\alpha}{\otimes}} B$ respectively. For brevity, we omit the construction of these topologies. As a major breakthrough, [1, Lemma 3.1] describes these spaces as inverse limits of corresponding tensor products $A_p \hat{\underset{\nu}{\otimes}} B_q$, $A_p \hat{\underset{\lambda}{\otimes}} B_q$, $A_p \hat{\underset{\max}{\otimes}} B_q$ and $A_p \hat{\underset{\min}{\otimes}} B_q$ respectively of Banach* algebras (firt one), Banach spaces (second one) and C^*-algebras (last two). There is an already existing concept of an admissible topology on $A \otimes B$. We show that for every admissible topology τ on $A \otimes B$, $\alpha \leq \tau \leq \gamma$ and if either A or B is commutative then $\varepsilon = \alpha = \tau = \gamma$. In fact, the property $\varepsilon = \alpha$ is equivalent to the commutativity of either A or B. The original form of the following theorem contains ten equivalent conditions. Here we curtail the statement. [1, Lemma 3.1] plays a crucial rôle in this theorem.

Theorem 2.8. *Let A, B be pro-C^*-algebras. The following are equivalent.*

(1) Either A or B is abelian.
(2) Any continuous pure state ω on $A \hat{\underset{\alpha}{\otimes}} B$ is of the form $\omega_1 \otimes \omega_2$, where ω_1, ω_2 are continuous pure states on A, B respectively.
(3) $A \hat{\underset{\varepsilon}{\otimes}} B = A \hat{\underset{\alpha}{\otimes}} B$ with $\varepsilon = \alpha$ and the natural ε-calibration on $A \hat{\underset{\varepsilon}{\otimes}} B$ is an m^*-calibration.
(3') The natural ε-calibration on $A \hat{\underset{\varepsilon}{\otimes}} B$ is an m^*-calibration.
(4) $b(A) \hat{\underset{\lambda}{\otimes}} b(B) = b(A) \hat{\underset{min}{\otimes}} b(B)$ with $\lambda = \| \cdot \|_{min}$.
(5) $A \hat{\underset{\varepsilon}{\otimes}} b(B) = A \hat{\underset{\alpha}{\otimes}} b(B)$ with $\varepsilon = \alpha$ and the natural calibration ε on $A \hat{\underset{\varepsilon}{\otimes}} b(B)$ is an m^*-calibration.

It should be empathetically noted that the condition "The natural ε-calibration on $A \hat{\underset{\varepsilon}{\otimes}} B$ is an m^*-calibration." and similar other as in (3) are essential and cannot be dropped.

Proposition 2.9. *Let A_1, B_1, A_2, B_2 be pro-C^*-algebras, $\phi_i : A_i \to B_i$, $(i = 1, 2)$, be continuous completely positive maps. Then $\phi_1 \otimes \phi_2$ is $\alpha - \alpha$ and $\gamma - \gamma$ continuous and extends continuously to respective completions as completely positive maps.*

A pro-C^*-algebra A is said to be *nuclear* if A_p is a nuclear C^*-algebra for every $p \in S(A)$, equivalently, [1, Proposition 4.1] it is an inverse limit of nuclear C^*-algebras with the surjective projection maps. We shall mention only a few results. A pro-C^*-algebra A is nuclear if any one of the following holds.

(1) A is commutative.

(2) A is type I.

(3) A is finite dimensional.

A pro-C^*-algebra A is nuclear iff each $M_n(A)$ is so. For pro-C^*-algebras A, B, the identity map $A \otimes B \to A \otimes B$ extends to a continuous *-homomorphism $\psi : A \underset{\gamma}{\hat{\otimes}} B \to A \underset{\alpha}{\hat{\otimes}} B$. However, A is a nuclear pro-C^*-algebra iff for every pro-C^*-algebra B, the corresponding extension ψ is a homeomorphism. An admissible topology τ on $A \otimes B$ is *faithful* if the map $\iota_\tau : A \underset{\tau}{\hat{\otimes}} B \to A \underset{\varepsilon}{\hat{\otimes}} B \subset B(A^*, B^*)$, $\iota_\tau(z) = (x' \otimes y')(z)$, $(x' \in A^*, y' \in B^*)$, is one-one. The injective tensorial topology α is faithful [1, Lemma 4.3]. Again the following is curtailed by omitting σ-C^*-algebra case.

Theorem 2.10. *([1, Theorem 4.2 and 4.5]) For a pro-C^*-algebra, the following are equivalent.*

(1) A is nuclear.

(2) For all pro-C^*-algebras B, $A \underset{\alpha}{\hat{\otimes}} B = A \underset{\gamma}{\hat{\otimes}} B$ with $\alpha = \gamma$.

(3) For all C^*-algebras B, $A \underset{\alpha}{\hat{\otimes}} B = A \underset{\gamma}{\hat{\otimes}} B$ with $\alpha = \gamma$.

(4) For all pro-C^*-algebras (respectively C^*-algebras) B, there is only one admissible pro-C^*-topology on $A \otimes B$.

(5) $b(A)$ is a nuclear C^*-algebra.

Going a bit further in the paper, [1, Theorem 4.11] states that A is a nuclear pro-C^*-algebra iff for every pro-C^*-algebra B, every continuous complete state map from A to the strong dual B^* can be approximated in the simple weak* convergence by complete state maps of finite ranks from A to B^*.

Theorem 2.11. *([1, Theorem 4.13]) For a pro-C^*-algebra, the following are equivalent.*

(1) A is nuclear.

(2) For every C^*-algebra B, for every continuous complete state map $\phi : A \to B^*$ and for every $p \in S(A, \phi) = \{p \in S(A)$: there is $K > 0$ such that $\|\phi(x)\| \leq Kp(x)$ for all $x \in A\}$, there exists a net $\{\phi_j\}$ of continuous complete state maps $\phi_j : A \to B^*$ of finite ranks such that

(a) $\phi = \lim_j \phi_j$ in simple weak* convergence.

(b) $p \in S(A, \phi_j)$ for all j.

We end the discussion on [1] with the following result about the multipliers of the Pedersen ideal. For a pro-C^*-algebra A the multipliers are automatically continuous. We denote the set of all multipliers (l, r) on A by $M(A)$. If $M(A)$ is nuclear, then so is A [1, Corollary 4.8(2)], however, the converse fails even in the case of the C^*-algebra. For an infinite dimensional Hilbert space H, the algebra $K(H)$ of all compact operators on H is nuclear but $M(K(H)) = B(H)$ fails to be nuclear.

Theorem 2.12. *([1, Theorem 5.1]) Let A be a pro-C^*-algebra. The C^*-algebra $b(M(A))$ is isometrically *-isomorphic to the C^*-algebra $M(b(A))$. Thus $M(A)$ is a nuclear C^*-algebra iff $b(A)$ and the generalized Calkin algebra $M(b(A))/b(A)$ are nuclear C^*-algebra.*

We omit the discussion on the relation between Grothendieck's nuclearity of a pro-C^*-algebra as a locally convex space and the C^*-nuclearity discussed in the final section of the paper.

The concept of enveloping pro-C^*-algebra was introduced and investigated mainly in [Bro71], [Ino71] and [Fra81]. In [3], we characterized, investigated and gave abundance of examples of those complete lmc *-algebra A for which the enveloping pro-C^*-algebra $\mathscr{E}(A)$ turns out to be a C^*-algebra. For a Banach* algebra A, [BD73b] defines the greatest (automatically continuous) C^*-seminorm $m(\cdot)$ on A. This has been obtained by $m(x) = \sup\{\|\pi(x)\| : \pi$ is a representation of $A\} = \sup\{\|\pi(x)\| : \pi$ is an irreducible representation of $A\}$, $(x \in A)$. The Hausdorff completion of $(A, m(\cdot))$ is a C^*-algebra, which was later on baptised as the *enveloping C^*-algebra of A*. In [3], we fix a complete lmc *-algebra A and also $P \in \mathscr{P}^*$. A net (e_γ) in A is said to be a *bounded approximate identity (bai)* if $p(e_\gamma) \le 1$ for every $p \in P$ and $xe_\gamma \to x$, $e_\gamma x \to x$ for every $(x \in A)$. Throughout, we assume that A has a bai and fix a bai (e_γ) in A. $K(A)$ denotes the set of all continuous algebra *-semonorms p on A such that $p(e_\gamma) \le 1$ for every γ. $R(A)$ and $R'(A)$ respectively denote the sets of all continuous *-representations and all continuous topologically irreducible *-representations $\pi : A \to B(H_\pi)$, where H_π is a Hilbert space. For $p \in K(A)$, $R_p(A) = \{\pi \in R(A) :$ there is $k > 0$ such that $\|\pi(x)\| \le kp(x)$ for all $x \in A\}$, $R'_p(A) = R'(A) \cap R_p(A)$. The first observation is that

$$R(A) = \cup_{p\in P} R_p(A) = \cup_{p\in K(A)} R_p(A) \text{ and } R'(A) = \cup_{p\in P} R'_p(A) = \cup_{p\in K(A)} R'_p(A).$$

For $p \in K(A)$, $r_p(x) = \sup\{\|\pi(x)\| : \pi \in R_p(A)\} = \sup\{\|\pi(x)\| : \pi \in R'_p(A)\}$ is a continuous C^*-seminorm on A [Fra81, Lemma 4.1]. Consequently, the *-radical $\operatorname{srad}(A)$ of A is given by $\cap_{p\in P} N_{r_p} = \cap_{p\in K(A)} N_{r_p}$ is a two sided *-ideal of A and $r_p(x + \operatorname{srad}(A)) = r_p(x)$, $(x + \operatorname{srad}(A) \in A/\operatorname{srad}(A))$, defines a continuous C^*-seminorm on $A/\operatorname{srad}(A)$. In fact, $\{r_p : p \in K(A)\}$ is the separating family of all continuous C^*-seminorms on $A/\operatorname{srad}(A)$. The completion of $A/\operatorname{srad}(A)$ with the topology given by $\{r_p : p \in K(A)\}$ is a pro-C^*-algebra, called the *enveloping pro-C^*-algebra of A* and is denoted by $(\mathscr{E}(A), \tau)$. We say that a A has a C^*-enveloping algebra if $(\mathscr{E}(A), \tau)$ is a C^*-algebra.

Theorem 2.13. *([2, Theorem 2.1]) The algebra A has a C^*-enveloping algebra iff A admits the greatest C^*-seminorm, (denoted by p_∞). In this case, $p_\infty(x) = \sup\{r_p(x) : p \in P\} = \sup\{\|\pi(x)\| : \pi \in R(A)\} = \sup\{\|\pi(x)\| : \pi \in R'(A)\}$, $(x \in A)$; and $(\mathscr{E}(A), \tau) = (A/N_{p_\infty})^\sim$, the completion of A/N_{p_∞} in the norm $\|x + N_{p_\infty}\|_\infty = p_\infty(x)$.*

A Q-algebra (that is, an algebra in which the set of all quasiinvertible elements is open) has a C^*-enveloping algebra. There are algebras with C^*-enveloping algebra which fail to be spectrally bounded, and hence, are not Q. However, a hermitian algebra with C^*-enveloping algebra is a Q-algebra. For $x \in A$, the *hermitian spectral radius of* x is defined to be $r^h(x) = \sup\{r(\pi(x)) : \pi \in R'(A)\}$, where $r(\pi(x))$ is the spectral radius of $\pi(x)$. Similar to the characterization of a Q-algebra in terms of spectral radius of an element we have the following.

Corollary 2.14. *([2, Corollary 2.5]) The algebra A has a C^*-enveloping algebra iff there is $p \in K(A)$ and $k > 0$ such that $r^h(x) \le kp(x)$ for all $x \in A$.*

A *algebra is called **-spectrally bounded (*sb)* if for every $x \in A$, the spectral radious of x^*x is finite, that is, $r(x^*x) < \infty$. In this case, we define $s(x) = r(x^*x)^{1/2}$. A *sb algebra admits the greatest C^*-seminorm $|\cdot|$, however, it need not be continuous. A is hermitian iff $|\cdot| = s(\cdot)$ iff $s(\cdot)$ is a C^*-seminorm. Thus a *sb A has a C^*-enveloping algebra iff A is hermitian and $s(\cdot)$ is a continuous C^*-seminorm on A. The remaining part of [2] has more than ten and about fifteen typical examples (categorized mainly into function algebras, Sobolev algebras, Segal algebras, topological algebras with bases Köthe sequence algebras). But the discussion requires a lot of technical terminology to be developed in terms of many spectral functions, GNS construction, equicontinuous families of positive linear functionals and lot more. So, we restrict the discussion up to this point only.

Next paper improves on [Fra91, Proposition 3.8] removing a number of conditions from the hypothesis and obtaining the autocontinuity based on algebraic condition.

Theorem 2.15. *([5]) Let E be a *-sb *-algebra, F be a barrelled pseudocomplete pro-C^*-algebra and $\phi : E \to F$ be a surjective *-homomorphism. Then the topology of F is normable and is in fact, a C^*-algebra.*

There are certain lemmas and other results improving results of [Fra91]. It is shown by means of examples that our conditions cannot be further weakened.

In the course of the discussion on [6], A will be a real algebra. The Gel'fand Mazur Theorem(s) is more a phenomenon than merely one theorem. [BD73b, Theorem 14.7] states some conditions under which a Banach algebra is $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$, the quaternions. Zalar [Zal95] obtained GMT for algebras having inner product and assuming relations with square of elements and norm. We obtained the C^*-analogue of the same and went on to obtain more GMTs. The following GMT is also related to the Froelich-Ingelstam-Smiley Theorem [Fro93].

Theorem 2.16. *([6, Theorem 1])*

Let $\|\cdot\|$ be a Pythagorean norm on a real *-algebra A.

(a) Assume that $\|\cdot\|$ satisfies at least one of the following.

(i) (1) A has identity, $\|\mathbf{1}\| = 1, \|a^*a\| \leq \|a\|^2$ for all $a \in A$, and (2) $a^*a = 0 \Rightarrow a = 0$, $(a \in A)$.

(ii) $\|a^*a\| = \|a\|^2$ for all $a \in A$.

Then A is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

(b) Suppose A is a complex *-algebra. Assume that at least one of the following holds.

(i) A has identity, $\|\mathbf{1}\| = 1$ and $\|a^*a\| \leq \|a\|^2$ for all $a \in A$.

(ii) $\|a^*a\| = \|a\|^2$ for all $a \in A$.

Then $A \cong \mathbb{C}$.

For a *-algebra A, $\operatorname{Sym} A = \{a \in A : a^* = a\}$.

Theorem 2.17. *([6, Theorem 2]) Let A be a real *-algebra with identity $\mathbf{1}$. Assume that*

$$a^*a = 0 \text{ implies } a = 0 \text{ for } a \in A. \tag{1}$$

Let $\|\cdot\|$ be a norm on $\operatorname{Sym} A$. Suppose for all $a, b \in \operatorname{Sym} A$ satisfying that $ab = ba$, that

$$\|a+b\|^2 + \|a-b\|^2 = 2\|a\|^2 + 2\|b\|^2.$$

Assume that at least one of the following holds.

(i) $\|\mathbf{1}\| = 1$, $\|a^2\| \leq \|a\|^2$, $(a \in \operatorname{Sym} A)$.

(ii) $\|a^2\| = \|a\|^2$ for all $a \in \operatorname{Sym} A$.

Then A is isomorphic to $\mathbb{R}$, $\mathbb{C}$ of $\mathbb{H}$.

Following Goodreal [Goo82], a *real C^*-algebra* is a real *-algebra with norm $\|\cdot\|$ satisfying $\|ab\| \leq \|a\|\|b\|$, $\|a^*a\| \leq \|a\|^2$ for all $a, b \in A$; and for each $a \in A$, $\mathbf{1} + a^*a$ is invertible in the unitization of A.

Theorem 2.18. *([6, Theorem 3])*

(a) Let A be a real C^*-algebra. Assume that at least one of the following holds.

(i) A has identity, $|h| = \pm h$ for all $h \in \operatorname{Sym} A$.

(ii) $\operatorname{Sym} A$ has no nonzero zero divisor.

Then A is isomorphic to $\mathbb{R}$, $\mathbb{C}$ or $\mathbb{H}$.

(b) Let A be a complex C^*-algebra satisfying at least one of the above (aii) *or* (ai) *or the following.*

(iii) A is unital and for each $h \in \operatorname{Sym} A$, e^{ih} has a convex spectrum.

Then A is isomorphic to $\mathbb{C}$.

An algebra with identity $\mathbf{1}$ is said to be *locally finite* if for each $h \in A$, the smallest algebra containing h and $\mathbf{1}$ is finite dimensional.

Theorem 2.19. *([6, Theorem 4]) Let A be a real Banach algebra with identity $\mathbf{1}$. Suppose that A has no nonzero zero divisor and A is locally finite. Then A is isomorphic to $\mathbb{R}$ or $\mathbb{C}$ or $\mathbb{H}$.*

In [9], we investigate the Sobolev Banach Algebras. In what follows $AC[0,1]$ denotes the algebra of all absolutely continuous complex valued functions on $[0,1]$ whereas $CBV[0,1]$ stands for the algebra of all complex valued continuous functions with bounded variation on $[0,1]$. Let $(A, \|\cdot\|)$ be any of the following Banach spaces with the respective norms.

(1) For $1 \le p < \infty$ and $m \in \mathbb{N}$, let

$$W^{m,p}[0,1] = \{f \in C^{m-1}[0,1] : f^{(m-1)} \in AC[0,1], \text{ and } f^{(m)} \in L^p[0,1]\}$$

with the norm $\|f\|_{m,p} = \sum\limits_{k=0}^{m-1} \frac{1}{k!}\|f^{(k)}\|_\infty + \frac{1}{m!}\left(\int_0^1 |f^{(m)}(t)|^p dt\right)^{\frac{1}{p}}$, $(f \in W^{m,p}[0,1])$.

(2) For $m \in \mathbb{N}$, let

$$W^{m,bv}[0,1] = \{f \in C^{m-1}[0,1] : f^{(m-1)} \in CBV[0,1] \text{ and } f^{(m)} \in L^p[0,1]\};$$

with the norm $\|f\|_{m,bv} = \sum\limits_{k=0}^{m-1} \frac{1}{k!}\|f^k\|_\infty + \frac{1}{m!}\operatorname{Var}_{[0,1]} f^{(m-1)}$, $(f \in W^{m,bv}[0,1])$.

Theorem 2.20. *([9, Theorem 1 and 2]) $W^{m,p}[0,1]$ as well as $W^{m,bv}[0,1]$ are semisimple commutative unital Banach algebras with the pointwise multiplication. The Gel'fand space of each of these algebras is homeomorphic to* $[0,1]$

The paper also initiates the investigation of the ideal theory of these algebras.

In [10], we obtain various multiplications on a two dimensional algebra up to isomorphism.

We now jump onto the final paper [11]. The others will be reported by our coauthors.

Let X be a compact Hausdorff space and $\mathcal{L}t(X)$ denote the algebra of all functions $f : X \to \mathbb{C}$ such that $\lim\limits_{t \to x} f(t)$ exists for every $x \in X$. Then $C(X) \subset \mathcal{L}t(X) \subset B(X)$, the algebra of all bounded complex-valued functions on X. It is shown that $\mathcal{L}t(X)$ is a commutative unital C^*-algebra with sup norm and its maximal ideal space $\triangle(\mathcal{L}t(X))$ is the disjoint union of X with itself. However, the topology on $\triangle(\mathcal{L}t(X))$ is strictly finer than the disjoint union topology.

3. My papers with Professor Bhatt

(1) *Complete positivity, tensor products and C^*-nuclearity for inverse limits of C^*-algebras*, Proc. Indian Acad. Sci. Math. Sci., **101**(3) (1991) 149–167.

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(11) *A curious Banach algebra: Banach algebra of functions with limits*, Math. Student, **85**(3-4) (2016)13–19.

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