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SOME RESULTS IN OPERATOR THEORY AND ANALYSIS

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1. Tribute

In 1976, when I was studding in M. A., I approached Subhashbhai with a question: "If a complex Banach algebra A with identity has a property that every element has singleton spectrum, is it isomorphic to $\mathbb{C}$, the field of complex numbers?". He said that in fact he proved a more general result that "if the spectrum of each element in a complex semi simple Banach algebra is convex, then it is isomorphic to $\mathbb{C}$" [B]. After that we were reading together basics of unbounded operators in a Hilbert space and we became good friend since then. With this, I could work in the theory of unbounded operators. In fact one of my papers "A joint spectral theorem for unbounded normal operators" [P] is based on his work of his Ph. D. Thesis. We were staying in the same hostel and we were discussing, what I read during a day, up to late night. We had very much tuning and bonding; usually we had the same thinking on many issues pertaining to the department. He inspired me many times and suggested topics of research. He was a man of versatile thinking.

2. joint Norms

Let H be a separable Hilbert space and $BL(H)$ be the Banach Algebra of all bounded linear operators on H. Let $T = (T_1, T_2, ..., T_n)$ be an n-tuple of operators in $BL(H)$.

For $1 \leq p < \infty$, consider joint norms

$$\|T\|_p = (\sum_{i=1}^{\infty} \|T_i\|^p)^{1/p}, \text{ where } \|\cdot\| \text{ is the operator norm on } BL(H) \tag{1}$$

$$|T|_p = \sup\{(\sum_{i=1}^{\infty} \|T_i x\|^p)^{1/p} : x \in H, \|x\| = 1\}. \tag{2}$$

For a natural number n, consider $BL(H)^n$, the product of n copies of $BL(H)$. For $S = (S_1, S_2, ..., S_n)$ and $T = (T_1, T_2, ..., T_n) \in BL(H)^n$, define $ST = (S_1T_1, S_2T_2, ..., S_nT_n)$. With this multiplication and with each of above norms, $BL(H)^n$ is a Banach algebra. Since $|T|_p \leq \|T\|_p \leq n^{1/p}|T|_p$, the joint norms $|\cdot|_p$ and $\|\cdot\|_p$ are equivalent. Also since $\|\cdot\|_p$ and $\|\cdot\|_q$ are equivalent for $p, q \geq 1$, the joint norms $\|\cdot\|_p$ and $|\cdot|_q$ are equivalent for $p, q \geq 1$. But for $n > 1$, none of these norms is a C^*-norm, however each of these norms is equivalent to the C^*-norm $\|T\|_\infty = \max\{\|T_i\| : i = 1, 2, ..., n\}$.

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Let $\mathfrak{C}(H)$ be the C^*-algebra of all compact operators on H. An operator $T \in \mathfrak{C}(H)$, is called a trace class operator if $\sum_{i=1}^{\infty} | < Tu_i, u_i > | < \infty$, where $\{u_i\}$ is an orthonormal basis of H. The set of all trace class operators is denoted by $\mathfrak{J}(H)$. In fact for $T \in \mathfrak{J}(H)$, the sum $Tr(T) = \sum_{i=1}^{\infty} < Tu_i, u_i >$ is independent of an orthonormal basis $\{u_i\}$ of H and it is known as the trace of T. $\mathfrak{J}(H)$ is a Banach algebra with the norm $c_1(T) = Tr(|T|)$, where $|T| = (T^*T)^{\frac{1}{2}}$ [Sa]. Consider $\mathfrak{J}(H)^n$, the product of n-copies of $\mathfrak{J}(H)$. For $T = (T_1, T_2, ..., T_n) \in \mathfrak{J}(H)^n$, let $\alpha_p(T) = (\sum_{i=1}^{n}(c_1(|T_i|))^p)^{1/p}$, $1 \le p < \infty$. Then for $1 \le p < \infty$, $\alpha_p(\cdot)$ is a norm on $\mathfrak{J}(H)^n$.

It is well known that the dual of $(\mathfrak{J}(H), c_1(\cdot))$ is $(BL(H), \|\cdot\|)$ and the dual of $(\mathfrak{C}(H), \|\cdot\|)$ is $(\mathfrak{J}(H), c_1(\cdot))$, under the usual identifications [Sa]. Following theorem extends these for $n > 1$ and for different norms [PB].

Theorem 2.1. *Let $1 < p < \infty$ and q be exponent conjugate of p (i.e. $1/p + 1/q = 1$). Then*

(a) The dual of $(\mathfrak{J}(H)^n,\ \alpha_q(\cdot))$ is isomertically isomorphic to $(BL(H)^n,\ \|\cdot\|_p)$.

(b) The dual of $(\mathfrak{C}(H)^n,\ \|\cdot\|_p)$ is isomertically isomorphic to $(\mathfrak{J}(H)^n,\ \alpha_q(\cdot))$.

For $T = (T_1, T_2, ..., T_n) \in \mathfrak{J}(H)^n$, consider $\beta_2(T) = Tr((\sum_{i=1}^{n} T_iT_i^*)^{1/2})$.

Next we consider similar problem for the norm $|\cdot|_2$ defined in (2) in the following.

Theorem 2.2. *The dual of $(\mathfrak{J}(H)^n, \beta_2(\cdot))$ is isomertically isomorphic to $(BL(H)^n, |\cdot|_2)$.*

Open problem: In general we may define β_p properly for $1 < p < \infty$, so that the dual of $(\mathfrak{J}(H)^n, \beta_q(\cdot))$ may be isometrically isomorphic to $(BL(H)^n, |\cdot|_p)$.

3. Unbounded Subnormal Operators

Let T be a linear operator (not necessarily bounded) defined on a subspace $\mathfrak{D}(T)$ of a Hilbert space H. A closed subspace M of H is called invariant under T if $T(M \cap \mathfrak{D}(T)) \subset M$. M is called reducing under T if $T(M\mathfrak{D}(T)) \subset M$; $T(M^\perp \cap \mathfrak{D}(T)) \subset M^\perp$ and $\mathfrak{D}(T) = [M \cap \mathfrak{D}(T)] + [M^\perp \cap D(T)]$.

A closed linear operator T defined on a dense subspace $\mathfrak{D}(T)$ of H is called normal if, $\mathfrak{D}(T) = \mathfrak{D}(T^*)$ and $\|T^*x\| = \|Tx\|$ for all $x \in \mathfrak{D}(T)$ (equivalently $\mathfrak{D}(T^*T) = \mathfrak{D}(TT^*)$ and $T^*Tx = TT^*x$ for all $x \in \mathfrak{D}(T^*T)$).

Note that a restriction of a normal operator T in a subspace M reducing under T is also normal with domain $\mathfrak{D}(T|M) = M \cap \mathfrak{D}(T)$.

A linear operator S defined on a dense subspace $\mathfrak{D}(S)$ of H to H is called subnormal if there is a Hilbert space K and a normal operator N defined on a dense subspace $\mathfrak{D}(N)$ of K such that H is a closed subspace of K; $N(\mathfrak{D}(S)) \subset H$ and $Sx = Nx$ for all $x \in \mathfrak{D}(S)$. Since a densely defined symmetric operator in a Hilbert space always has a self adjoint extension, a symmetric operator is a subnormal operator.

A normal extension $(N, \mathfrak{D}(N), K)$ of a subnormal operator $(S, \mathfrak{D}(S), H)$ is called minimal normal extension if for any normal extension $(N_1, \mathfrak{D}(N_1), K_1)$ of S such that K_1 is reducing under N, $\mathfrak{D}(S) \subset \mathfrak{D}(N_1) \subset \mathfrak{D}(N)$ and $N_1x = Nx$, $x \in \mathfrak{D}(N_1)$, then $K_1 = K$ and $N_1 = N$ (i.e. $\mathfrak{D}(N_1) = \mathfrak{D}(N)$ and $N_1x = Nx$ for all $x \in \mathfrak{D}(N_1)$).

For an operator T defined in H, elements in $C^\infty(T) = \bigcap_{n=1}^{\infty} \mathfrak{D}(T^n)$ are called C^∞ vectors of T. Suppose a normal extension of $(N, \mathfrak{D}(N), K)$ of a subnormal operator $(S, \mathfrak{D}(S), H)$ such that the set $\{N^{*i}N^jx : x \in C^\infty(S), i, j = 1, 2, 3, ...\}$ is linearly dense in K. Stochel and Szafranic called such extension as a minimal normal extension [SS] (it was not proved). Second half of the following theorem shows that in fact it is minimal defined as above [BP1].

Theorem 3.1. *(a) Every subnormal operators in H has a minimal normal extension.*

(b) Let S be a subnormal operator defined on a dense domain $\mathfrak{D}(S)$ in H and $(N, \mathfrak{D}(N), K)$ be a normal extension of $(S, \mathfrak{D}(S), H)$. Let $\mathfrak{D}$ be the linear span of the set $\{N^{*i}N^jx : x \in C^\infty(S), i, j = 1, 2, 3...\}$. Then

(1) if $\mathfrak{D}$ is dense in K, then N is a minimal normal extension of S.

(2) if N is a is a minimal normal extension of S and $\mathfrak{D}(N) = \mathfrak{D} + (\mathfrak{D}(N) \cap \mathfrak{D}^\perp)$, then $\mathfrak{D}$ is dense in K.

The spectrum of an operator T defined in H is the set $\sigma(T) = \{\lambda \in \mathbb{C} : T - \lambda I$ does not have a bounded inverse$\}$.

Following is a generalization of a result of bounded subnormal operator to unbounded subnormal operator [BP1].

Theorem 3.2. *Let S be a subnormal operator in H with domain $\mathfrak{D}(S)$ and N be a minimal normal extension of S. Then $\sigma(N) \subset \sigma(S)$.*

Example: Consider $U = \{z \in \mathbb{C} : |z| < 1\}$ and $\Gamma = \{z \in \mathbb{C} : |z| = 1\}$. For a measurable function φ on Γ consider $\mathfrak{D}_\varphi = \{f \in H^2(U) : \varphi f \in L^2(\Gamma)\}$. Define T_φ in $H^2(U)$ with domain $\mathfrak{D}_\varphi$ as $T_\varphi(f) = P(\varphi f)$, $f \in \mathfrak{D}_\varphi$, where $P : L^2(\Gamma) \to H^2(U)$ is the orthogonal projection. This T_φ is called Toeplitz operator, if φ is analytic on U, then T_φ is called analytic Toeplitz operator. Such analytic Toeplitz operator T_φ is a subnormal operator with a normal extension M_φ with domain $\mathfrak{D}(M_\varphi) = \{f \in L^2(\Gamma) : \varphi f \in L^2(\Gamma)\}$ defined by $M_\varphi f = \varphi f$, $f \in \mathfrak{D}(M_\varphi)$.

4. Multiplicativity factors of seminorms on involutive algebras

Let A be an algebra over the field of complex numbers. Let p be a seminorm on A (i.e. $p : A \to [0, \infty)$ such that $p(0) = 0$, $p(x + y) \le p(x) + p(y)$ and $p(\lambda x) = |\lambda|p(x)$ for all $x, y \in A$ and a scalar λ). p is said to have an M-factor m (respectively a Q factor q) if $p(xy) \le mp(x)p(y)$ (respectively $p(x^2) \le qp(x)^2$ for all $x, y \in A$. M-factors and Q-factors of seminorms are studied in [AG1], [AG2].

A seminorm p on an involutive complex algebra A is said to have an M^*-factor m (respectively a Q^* factor q) if $p(x^*y) \leq mp(x)p(y)$(respectively $p(x^*x) \leq qp(x)^2$ for all $x, y \in A$. s is called a *-factor of p if $p(x^*) \leq sp(x)$ for all $x \in A$. In [BP2], M^*-factors, Q^*-factors and *-factors of seminorms are studied.

Let A be an involutive algebra over the field of complex numbers and p be a seminorm on A. If p is has a C^* property: $p(x^*x) = p(x)^2$ for all $x \in A$, then p is *-invariant i.e. $p(x^*) = p(x)$ for all $x \in A$ [Se] and p is submultiplicative i.e. 1 is an M factor of p [BD]. If p satisfies C^*-inequality $k_1p(x)^2 \leq p(x^*x) \leq k_2p(x)^2$, for all $x \in A$, then p has an M-factor, p has a *-factor and p is equivalent to a C^*-seminorm.

Note that there is a commutative involutive algebra A with identity and a seminorm p on A such that (a) $p(x^2) \leq p(x^*x)$ for all $x \in A$, (b) p is *-invariant and (c) $p(1) = 1$ but p does not have an M-factor [BP2].

Theorem 4.1. *Let A be an involutive algebra over the field of complex numbers. Then*

(a) If a seminorm p on A has an M^*-factor, then it has a Q^*-factor.

*(b) Suppose a seminorm p on A has a *-factor. Then it has a Q^*-factor if and only if it has an M-factor.*

Let $\alpha = \sup\{p(x^*y) : x, y \in A \text{ and } p(x) = p(y) = 1\}$ and $\beta = \sup\{p(x^*) : x \in A,\ p(x) = 1\}$.

Theorem 4.2. *Let A be an involutive algebra over the field $\mathbb{C}$ and p be a seminorm on A. Then*

*(a) p has both *-factor and M^*-factor if and only if ker(p) $= \{x \in A : p(x) = 0\}$ is a *-ideal in A and α, β are finite.*

*(b) Suppose ker(p) is a *-ideal in A. Then*

(i) $\lambda > 0$ is an M^*-factor of p if and only if $\lambda \geq \alpha$ and

*(ii) $\lambda > 0$ is a *-factor of p if and only if $\lambda \geq \beta$.*

*(c) Suppose ker(p) is a *-ideal in A and $\alpha = 0$. Then λ is an M^*-factor of p iff $\lambda > 0$.*

Theorem 4.3. *Let A be an involutive algebra over the field of complex numbers and p be a norm on A. Then*

(a) (i) p has an M^*-factor if and only if α is finite.

*(ii) p has a *-factor if and only if β is finite.*

(b) (i) $\lambda > 0$ is an M^*-factor of p if and only if $\lambda \geq \alpha$.

*(ii) $\lambda > 0$ is a *-factor of p if and only if $\lambda \geq \beta$.*

Theorem 4.4. *Let A be a finite dimensional involutive algebra over the field $\mathbb{C}$.*

*(a) Let p be a norm on A. Then p has both a *-factor and an M^*-factor.*

*(b) Let p be a seminorm on A. Then ker(p) is a *-ideal in A if and only if p has both a *-factor and an M^*-factor.*

5. Ascoli-Arzela theorem for differentiable functions

Consider the Banach algebra $C([a,b])$ of scalar valued continuous functions on $[a,b]$ with sup norm $\|\cdot\|_\infty$ defined as $\|f\|_\infty = \sup\{|f(x)| : x \in [a,b]\}$.

A subset K of $C([a,b])$ is called equicontinuous if for $\epsilon > 0$, there is a $\delta > 0$ such that $|f(x) - f(y)| < \epsilon$ for all $f \in K$ and for all $x, y \in [a,b]$ with $|x - y| < \delta$.

A closed and bounded subset K of $C([a,b])$ is compact if and only if K is equicontinuous. It is the well-known Ascoli-Arzela theorem. Now consider $C^1([a,b])$, the Banach algebra of all scalar valued continuously differentiable functions on $[a,b]$, with the norm $\|\cdot\|$ defined by $\|f\|= \|f\|_\infty + \|f'\|_\infty$, $f \in C^1([a,b])$. Ascoli-Arzela type theorem for a subset of $C^1([a,b])$ is proved in [BPT].

A subset K of $C^1([a,b])$ is called equidifferentiable at $x_0 \in [a,b]$ if for $\epsilon > 0$, there is a $\delta = \delta(x_0, \epsilon) > 0$ such that $|\frac{f(x)-f(x_0)}{x-x_0} - f'(x_0)| < \epsilon$ for all $f \in K$ and for all $x \in [a,b]$ with $|x - x_0| < \delta$. $K \subset C^1([a,b])$ is called (uniformly) equidifferentiable on $[a,b]$ if in above, a δ can be chosen independent of $x_0 \in [a,b]$.

Theorem 5.1. *A closed and bounded subset K of $C^1([a,b])$ is compact iff $\lim_{h\to 0} \|f_h - f\| = 0$, uniformly over f in K, where $f_h(x) = f(x+h)$ and $\|\cdot\|$ is the C^1-norm on $C^1([a,b])$.*

Theorem 5.2. *A closed and bounded subset K of $C^1([a,b])$ is compact if and only if K is (uniformly) equidifferentiable on $[a,b]$.*

My Research Papers with Professor Bhatt

(1) *On unbounded subnormal operators*, Proc. Indian Acad. Sci. (Math Sci.) 99 (1989), 85-92.

(2) *On seminorms with multiplicativity factors on involutive algebras*, Math. Today 16 (1998), 17-24.

(3) *An Ascoli-Arzela theorem for differentiable functions*, (jointly with Thomas Mathew), Math. Stud. 56 (1988), 184-188.

(4) *A note on Joint Norms*, Indian Jour. Pure. Appl. Math. 15 (11) (1984) 1199-1205.

References

[AG1] R. Arens and M. Goldberg, *Multiplicativity factors for seminorms*, Jour. Math. Anal. Appl. 146 (1990) 469-481.

[AG2] R. Arens and M. Goldberg, *Quadratic seminorms and Jordan structures on algebras*, Linear Algebra and Appl. 181 (1993) 269-278.

[B] S. J. Bhatt, *Spectrally convex Banach Algebras*, Indian J. Pure Appl. Math. 10(6) (1979) 726-730.

[BD] S. J. Bhatt and H. V. Dedania, *On seminorms, spectral radius and Ptaks spectral function in Banach Algebras*, Indian Jour. Pure. Appl. Math. 27 (1996) 551-556.

[BP1] S. J. Bhatt and A. B. Patel, *On unbounded subnormal operators*, Proc. Indian Acad. Sci. (Math Sci.) 99 (1989), 85-92.

[BP2] S. J. Bhatt and A. B. Patel, *On seminorms with multiplicativity factors on involutive algebras*, Math. Today 16 (1998), 17-24.

[BPT] S. J. Bhatt, A. B. Patel and Thomas Mathew, *An Ascoli-Arzela theorem for differentiable functions*, Math. Stud. 56 (1988), 184-188.

[P] A. B. Patel, *A joint spectral theorem for unbounded normal operators*, Jour. Australian Math. Soc. (series A) 34 (1983) 203-213.

[PB] A. B. Patel and S. J. Bhatt, *A note on Joint Norms*, Indian Jour. Pure. Appl. Math. 15 (11) (1984) 1199-1205.

[Sa] S. Sakai, C*-algebras and W*-algebras, Springer-Verlag, Berlin (1971).

[Se] Z. Sebestyn, A C* seminorm is automatically submultiplicative, Period Math. Hunger. 10 (1979) 1-8.

[SS] J. Stochel and F. Szafranich, Normal extensions of Unbounded Operators, J. Operator Theory 14 (1985) 31-55.

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