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FRÉCHET ALGEBRAS, FORMAL POWER (LAURENT) SERIES, AND AUTOMATIC CONTINUITY

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1. Tribute

After completing my postgraduate study in 1997 and having cleared "National Eligibility Test", conducted by CSIR-UGC for Junior Research Fellowship and Eligibility for Lectureship in 1996, I joined Nirma Institute of Technology as a visiting lecturer, with a goal to get a permanent position there. I was really not so keen to pursue a full-fledged research program, but then I was inspired to take up the fellowship opportunity. Actually, one fine day I met Ex-Prof. Darshan Singh Basan, and he advised me to meet Professor Subhash Bhatt without looking for a place here and there. In December 1997, I met Professor Subhash Bhatt for the maiden time to discuss possibility of pursuing research under his able advice, and finally, I took admission in January 1998. Thus I began my research journey with him, and we have three papers. My third paper with him has been central to my further research, and is turned out to be a foundation work to solve several long standing, prestigious problems in automatic continuity theory. Thus, in 2020, when I look back in the past, I realize that how important it was for me to study algebras of power series in this paper, in order to solve certain fundamental problems in Fréchet algebras between 2005-2020. In fact, quite a couple of times in the past, he also appreciated my efforts of establishing remarkable, qualitative conjectures rather than producing results in quantity. Thus, I am very much grateful to him for introducing me theories of automatic continuity and Fréchet algebras for research. May ALMIGHTY GOD give his soul an eternal peace, and the courage to his wife, Sujataben and his daughter, Shreema to bear his absence. My deepest and heartfelt sympathies to both of them.

2. Our Research Work

Our research is mostly in the general theory of commutative Banach and Fréchet algebras, Fréchet algebras with a power (resp., Laurent) series generator, and Fréchet algebras of power series, with a few applications to automatic continuity theory in Fréchet algebras. Throughout the article, "algebra" will mean a complex, commutative algebra with identity unless otherwise

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specified. A *Banach algebra* is a complete, normed algebra A whose topology is induced by a submultiplicative norm $\|\cdot\|$. A *Fréchet algebra* is a complete, metrizable locally multiplicatively convex algebra A whose topology τ may be defined by an increasing sequence $(p_m)_{m\geq 1}$ of submultiplicative seminorms. We may refer to τ as "the Fréchet topology of A" in the following. The principal tool for studying Fréchet algebras is the Arens-Michael representation, in which A is given by an inverse limit of Banach algebras A_m (see [M, §5] or [P1, §2]). A Fréchet algebra A is called a uniform Fréchet algebra if for each $m \in \mathbb{N}$ and for each $x \in A$, $p_m(x^2) = p_m(x)^2$.

3. Fréchet algebras and formal Laurent series

Our second paper (jointly with H. V. Dedania), was published in 2002. A Fréchet algebra A has a Laurent series generator (briefly: L. s. g.) x, if (a) A is topologically generated by $\{x, x^{-1}\}$ for an invertible element x; and (b) for each $y \in A$, $y = \sum_{n\in\mathbb{Z}} \lambda_n x^n$, where $\sum_{n\in\mathbb{Z}} |\lambda_n| p_m(x^n) < \infty$ for all $m \in \mathbb{N}$. In Section 2 of (2), several examples of Fréchet algebras with a L.s.g. are discussed; e.g., Wiener-Fréchet algebras $\mathcal{W}(\Gamma_r, W)$ (in particular, $C^\infty(\Gamma)$, Γ the unit circle), various annulus algebras and Beurling-Fréchet algebras $\ell^1(\mathbb{Z}, W)$ for an increasing sequence $W = (\omega_m)_{m\geq 1}$ of weights on $\mathbb{Z}$. Though there are several functional analytic characterizations of holomorphic function algebras on simply connected planar domains, the case of annulus algebra appears to be treated for the first time in this paper. If p is a non-zero Laurent series seminorm on a Fréchet algebra A with a L.s.g., then p is a norm. Also, such an A has the unique expression property (briefly: UEP) if and only if its topology is generated by a sequence of Laurent series norms. In fact, such an A having the UEP is homeomorphically isomorphic to the Beurling-Fréchet algebra $\ell^1(\mathbb{Z}, W) = \bigcap_{m\geq 1}^{\infty} \ell^1(\mathbb{Z}, \omega_m)$. The following lemma explains the Arens-Michael representation of a Fréchet algebra A with a L.s.g. (see Lemma 3.7 of (2)).

Lemma 3.1. *Let A be a Fréchet algebra generated by an invertible element x. Then the following are equivalent.*

(a) *A is a Fréchet algebra with a L.s.g. x and having the UEP.*

(b) *There exists an inverse limit sequence*

$$A_1 \xleftarrow{\pi_1} A_2 \xleftarrow{\pi_2} A_3 \xleftarrow{\pi_3} A_4 \xleftarrow{\pi_4} \cdots$$

of Banach algebras A_m with L.s.g.s x_m and having the UEP such that $A = \varprojlim A_m$. □

We have the following main theorem from (2) among other results of some independent interest.

MAIN THEOREM. *Let A be a Frechet algebra with a L.s.g. x and having the UEP. Then the following holds.*

(a) *If the spectrum $sp_A(x)$ of x is open, then A is homeomorphically isomorphic to $H(\Gamma(r_2, r_1))$ for some $0 \leq r_2 < r_1 \leq \infty$. Further,*

(i) $0 < r_2$ *if and only if zero belongs to the interior of* $\mathbb{C} \setminus sp_A(x)$*; and*

(ii) $r_1 < \infty$ *if and only if* $sp_A(x)$ *is bounded.*

(b) *If the interior of* $sp_A(x)$ *is empty, then there exists* $r > 0$ *and a sequence* $W = (\omega_m)_{m\geq 1}$ *of weights on* $\mathbb{Z}$ *such that* A *is homeomorphically isomorphic to* $\mathcal{W}(\Gamma_r, W)$*. Further, if the generator* x *satisfies the condition (*), then* A *is homeomorphically isomorphic to* $C^\infty(\Gamma)$*.*

(c) *If for each* $m \in \mathbb{N}$, $p_m(x^n) = p_m(x)^n$ $(n > 0)$ *and* $p_m(x^n) = p_m(x^{-1})^{-n}$ $(n < 0)$*, then* A *is homeomorphically isomorphic to one of the Fréchet algebras* $H(\Gamma(r_2, r_1))$, $H(\Gamma[r_2, r_1))$, $H(\Gamma(r_2, r_1])$ *or* $H(\Gamma[r_2, r_1])$*. Further,* $sp_A(x)$ *is compact implies that* A *is the Banach algebra* $H(\Gamma[r_2, r_1])$*.*

(d) *If* A *is a uniform Fréchet algebra, then* A *is homeomorphically isomorphic to* $H(\Gamma(r_2, r_1))$*.* □

The notion of Banach algebras with a L.s.g. has been used to generalize the results, obtained in Sections 4 and 6, at the level of an operator-valued Baker function [DGP, Proposition 8.1]. Thus this work has an application in algebraic/differential geometry.

4. Fréchet algebras and formal power series

Our maiden paper was published in 2001. In this paper, we proved the Banach-algebra-analogue of Lemma 4.2 below. Further, A_1 is inverse closed in A if and only if $\mathrm{sp}_A(x)$ is a closed disc. This quickly gives Taylor series analogues of the classical theorems of Wiener and Levy on absolutely convergent Fourier series. I am happy to know that Professor Subhash Bhatt's daughter, Shreema recently referred to the work given in this paper.

An element x in a Fréchet algebra A is a power series generator (briefly: p.s.g.) for A if each $y \in A$ is of the form $y = \sum_{n=0}^{\infty} \lambda_n X^n$, $\lambda_n \in \mathbb{C}$, such that $\sum_{n=0}^{\infty} |\lambda_n| p_m(x^n) < \infty$ for all m. Let $\mathcal{F}$ be the Fréchet algebra of all formal power series $f = \sum_{n=0}^{\infty} \lambda_n X^n$ $(\lambda_n \in \mathbb{C})$ in an indeterminate X with the weak topology τ_c defined by the coordinate projections $\pi_k : \mathcal{F} \to \mathbb{C}, k \in \mathbb{Z}^+$, where $\pi_k(f) = \lambda_k$. A defining sequence of seminorms for $\mathcal{F}$ is (p_m), $p_m(\sum_{n=0}^{\infty} \lambda_n X^n) = \sum_{n=0}^{m} |\lambda_n|$. A *Fréchet algebra of power series* (briefly: FrAPS) is a subalgebra A of $\mathcal{F}$ such that (A, τ) is a Fréchet algebra containing X, and such that the inclusion $(A, \tau) \hookrightarrow (\mathcal{F}, \tau_c)$ is continuous. This paper is concerned with the following two questions on FrAPS A: (i) When is X a p.s.g. for A? (ii) When is A isomorphic to an inverse limit of Banach algebras of power series?

Our definition of a Fréchet algebra with a p.s.g. is motivated by Allan's definition of a Banach algebra with a p.s.g. [A1]. Banach algebras of power series (briefly: BAPS) have already been established as an important aspect of contemporary Banach algebra theory. Though FrAPS have been considered earlier by Loy ([L1, L2]); in last 25 years, they – and more generally, the power series ideas in general Fréchet algebras – have acquired significance in understanding the structure of a Fréchet algebra [A2, D, DPR, P1-7, R]. For examples of FrAPS, we refer to Section 1 of (3); in particular, $\mathcal{F}$, Beurling-Fréchet algebras $\ell^1(\mathbb{Z}^+, W)$ for an increasing sequence $W = (\omega_m)_{m\geq 1}$ of weights on $\mathbb{Z}^+$, the Banach algebra $A^+(D)$ (D the closed unit disc) of functions in the disc algebra $A(D)$ with absolutely convergent Taylor series on the unit circle Γ, the Fréchet algebra $\mathrm{Hol}(U)$ of all holomorphic functions on the open unit disc U, the Fréchet algebra $\mathbb{E}$ of all entire functions, and the algebra $A^\infty(\Gamma)$ (a closed subalgebra of $C^\infty(\Gamma)$). We have the following main theorem (Theorem 2.1 of (3)) among other results of some independent interest.

Theorem 4.1. *Let A be a FrAPS in an indeterminate X. Suppose that X is a p.s.g. for A. Then A is either $\mathcal{F}$ or the Beurling-Fréchet algebra $\ell^1(\mathbb{Z}^+, W)$ for an increasing sequence $W = (\omega_m)_{m\geq 1}$ of weights on $\mathbb{Z}^+$.* □

An element x in a Fréchet algebra A generates a cyclic basis if each $y \in A$ can be uniquely expressed as $y = \sum_{n=0}^{\infty} \lambda_n x^n$ ($\lambda_n \in \mathbb{C}$). A seminorm p on a FrAPS A having a p.s.g. x is a power series seminorm if $p(\sum_{n=0}^{\infty} \lambda_n X^n) = \sum_{n=0}^{\infty} |\lambda_n| p(X^n)$ for all $f = \sum_{n=0}^{\infty} \lambda_n X^n \in A$. To prove the above theorem, we need the following lemma (see Lemma 2.2 of (3)).

Lemma 4.2. *Let A be a singly generated Fréchet algebra with a cyclic basis generated by x. Then there exists a dense Fréchet subalgebra A_1 of A such that: (i) A_1 is continuously embedded in A; (ii) A_1 is a Fréchet algebra with a p.s.g. x; and (iii) A_1 is a Banach algebra provided that A is a Banach algebra.* □

We remark that if $(A, \|\cdot\|)$ is a normed algebra with a p.s.g. x (defined analogously), then its completion $\tilde{A}$ need not be a Banach algebra with a p.s.g. x. For example, let $(A(D), \|\cdot\|_\infty)$ be the disc algebra, and let $A^+(D)$ be the subalgebra of $A(D)$, then z is a p.s.g. for the normed algebra $(A^+(D), \|\cdot\|_\infty)$ containing the polynomials in z. Then, by the Mergelyan Theorem, the completion of $(A^+(D), \|\cdot\|_\infty)$ is the whole $A(D)$, and clearly, z is not a p.s.g. for the disc algebra $A(D)$ (see Remark 2.4 of (3)). For the subalgebras of $\mathcal{F}$, a power series norm on a normed algebra with a p.s.g. is a necessary and sufficient condition for the completion to be a BAPS with a p.s.g. (see Proposition 2.5 of (3)). Next, a seminorm p on a FrAPS A is closable if for any p-Cauchy sequence (f_k) in A, $f_k \to 0$ in τ_c implies that $p(f_k) \to 0$. Also, we define p to be of type (E) if for $k \in \mathbb{Z}^+$, there exists $c_m > 0$ such that $|\pi_k(f)| \leq c_k p(f)$ for all $f \in A$ [L2]. A seminorm of type (E) is a norm. We have the following gripping proposition (see Proposition 3.1 of (3)), which turns out to be very important to study certain automatic continuity problems for FrAPS (see Corollary 5.4 and Theorem 5.5 below).

Proposition 4.3. *Let A be a FrAPS. Let p be a continuous submultiplicative seminorm on A. Let $kerp = \{f \in A : p(f) = 0\}$. Let A_p be the completion of $A/kerp$ in the norm $\|f + kerp\|_p =$*

$p(f)$. Then the following are equivalent. (i) p is a norm and A_p is a BAPS. (ii) p is closable and of type (E). □

As a corollary, we obtain a crucial result (Corollary 3.2 of (3)); the several-variable-analogues of Proposition 4.3 and Corollary 4.4 will be used when we shall study FrAPS in $\mathcal{F}_k$ (see Theorems 5.13 and 5.14 below).

Corollary 4.4. *Let $A = \varprojlim A_m$ be the Arens-Michael representation of a FrAPS A. Assume that each p_m is a norm. Then each A_m is a BAPS if and only if each p_m is closable and of type (E).* □

It is readily seen that a FrAPS A satisfies the condition (E) in [L2] (i.e., there is a sequence (c_k) of positive reals such that $(c_k^{-1}\pi_k)$ is equicontinuous) if and only if A admits a continuous norms of type (E) if and only if the topology of A is defined by a sequence of seminorms of type (E). Hence the above corollary gives the following from [L1, L2] (see Corollary 3.4 of (3)). We remark that Theorem 4.1, and Corollaries 4.4 and 4.5 were one of the starting points to establish the uniqueness of the Fréchet topology for *all* FrAPS [P1] (resp., FrAPS in $\mathcal{F}_k$ [P3]). Note that a Beurling-Fréchet algebra $\ell^1(\mathbb{Z}^+, W)$ defined by a sequence of weights is expressible as an inverse limit of a sequence of BAPS.

Corollary 4.5. *Let A be the Beurling-Fréchet algebra $\ell^1(\mathbb{Z}^+, W)$. Then the following hold. (i) A has a unique Fréchet space topology as a topological algebra. (ii) Every derivation on A is continuous. (iii) A surjective homomorphism $\phi : B \to A$ from a Fréchet algebra B is continuous.* □

Being an inverse limit of finite dimensional algebras, $\mathcal{F}$ is a nuclear Fréchet space. The following gives another class of such algebras, exhibiting a significant difference between Banach and Fréchet algebras at the level of uniform algebras with a p.s.g. This is in view of the facts that (i) there exists an infinite dimensional uniform Banach algebra with a p.s.g. [A1, Proposition 4]; and that (ii) a nuclear Banach space is finite dimensional (Dvoretzky-Rogers Theorem). In the following, the topological algebraic property of being a uniform algebra forces the linear topological property of nuclearity (see Theorem 3.6 of (3)).

Theorem 4.6. *Let A be a uniform Fréchet algebra with a p.s.g. Suppose that A is not a Banach algebra. Then A is a nuclear space.* □

This recaptures the classical results that the algebras $\mathrm{Hol}(U)$ and $\mathbb{E}$ are nuclear. More generally, it follows from [P, Theorem 6.1.3] that a Beurling-Fréchet algebra $\ell^1(\mathbb{Z}^+, W)$ is nuclear if and only if $\ell^1(\mathbb{Z}^+, W) = \ell^\infty(\mathbb{Z}^+, W)$. Thus $A^\infty(\Gamma)$ is nuclear. Next, we give a functional analytic description of the holomorphic function algebra on a simply connected planar domain in the following theorem (see Theorem 3.7 of (3)).

Theorem 4.7. *Let A be a Fréchet algebra with a p.s.g. x. Then the following hold.*

(i) *Either $sp_A(x)$ is totally disconnected or A is isomorphic to $\ell^1(\mathbb{Z}^+, W)$ for a sequence $W = (\omega_m)$ of weights on $\mathbb{Z}^+$.*

(ii) *Suppose that $sp_A(x)$ is not totally disconnected and A is semisimple. Then the following hold. (a) If A is a Q-algebra, and if x satisfies condition (*), then A is isomorphic to $A^\infty(\Gamma)$. (b) If A is not a Q-algebra, then A is isomorphic to either $Hol(U)$ or $\mathbb{E}$.* □

5. Impact of third paper in Automatic Continuity Theory

Having worked with Professor Subhash Bhatt for my doctoral degree, I joined BITS-Pilani as a lecturer. In 2004, I again met Professor Garth Dales (U.K.) in Delhi, and discussed my research with him for the maiden time. In December 2004, I got a golden opportunity to work at IIT Kanpur. I then decided to only concentrate on the long standing, prestigious problems, and am happy to convey that I am remarkably succeeded in this plan. Our third paper has been playing a significant role in my further research since 2002, as we shall see now.

My first goal was to establish the uniqueness of the Fréchet topology for *all* FrAPS. Johnson (U.K.) established the uniqueness of the complete norm for BAPS in 1967 [J]. Allan (U.K.) established the uniqueness of the Fréchet topology of $\mathcal{F}$ in 1972 [A2, p. 276]. Loy (Australia) established the uniqueness of the Fréchet topology for FrAPS satisfying the condition (E) in 1971 (we remark that $\mathcal{F}$ does not satisfy the condition (E)) [L2]. Thus, the problem of establishing the uniqueness of the Fréchet topology for *all* FrAPS, was open since 1971. We also remark that the topology of a FrAPS A satisfying (E) is defined by a sequence of norms...(*). In fact, this was the starting point of the work given in [P1]. So, if we show that: (i) the converse of (*) holds; i.e., a FrAPS A whose topology is defined by a sequence of norms, necessarily satisfies (E), and (ii) a FrAPS A is either $\mathcal{F}$ or the topology of A is defined by a sequence of norms, then we are done due to the results of Allan and Loy.

First, we recall Allan's result [A2, Theorem 2]: Let A be a Banach algebra and let $x \in A$. Then the following are equivalent: (a) there is a unital, injective homomorphism $\theta : \mathcal{F} \to A$ such that $\theta(X) = x$. (b) $x \in \text{Rad}A$, x is non-nilpotent, and has finite closed descent. We consider the problem in the reverse direction and investigate the following. Whether we can describe *all* those commutative Fréchet algebras which may be continuously embedded in $\mathcal{F}$ in such a way that they contain X (and thus, the polynomials). Thus, we have completely characterized FrAPS as follows (see [P1, Theorem 3.1]).

Theorem 5.1. *Let A be a commutative, unital Fréchet algebra. Suppose that A contains a non-nilpotent, closed maximal ideal M such that: (i) $\bigcap_{n\geq 1} \overline{M^n} = \{0\}$ and (ii) $dim(M/\overline{M^2}) = 1$. Then A is a FrAPS. The converse holds if the polynomials are dense in A.* □

Then, we have the following elementary, but crucial, theorem [P1, Theorem 3.3], generalizing Proposition 4 of [A2] and Proposition 7.8, proved by Husain in his book "Orthogonal Schauder Bases" [Dekker, New York, 1991].

Theorem 5.2. *Let A be a FrAPS. Then either A is $\mathcal{F}$ or the Fréchet topology of A is defined by a sequence (p_m) of norms.* □

As corollaries, we have the following curious characterizations of $\mathcal{F}$ as a Fréchet algebra (see [P1, Corollaries 3.4 and 3.5]). Note that by a *proper* seminorm we mean a seminorm that is not a norm.

Corollary 5.3. *Let A be a FrAPS. Then $A = \mathcal{F}$ if and only if the Fréchet topology of A is defined by a sequence (p_m) of proper seminorms.* □

The main points of Corollary 5.3 should be emphasized. It has the surprising consequence that for a Fréchet algebra A of power series to have its Fréchet topology defined by an increasing sequence (p_m) of proper seminorms is, in fact, an *algebraic* property. Thus the topological structure of A here determines the algebraic structure of A. This is totally in contrast to what we would normally like to examine when and how the algebraic structure determines the topological structure, in particular, the continuity aspect in automatic continuity theory. There is a further consequence, which says that $\mathcal{F}$ is the *only* Fréchet algebras of finite type among FrAPS since an Arens-Michael representation of $\mathcal{F}$ contains finite-dimensional algebras, and, if $A \neq \mathcal{F}$, then A_m, the completion of (A, p_m), cannot be finite-dimensional algebras for each m. In fact, we have the following

Corollary 5.4. *Let A be a FrAPS such that the polynomials are dense in A. Then $A \neq \mathcal{F}$ if and only if $A = \varprojlim A_m$, where each A_m is a BAPS.* □

The immediate consequence of Corollary 5.4 is: $A\,(\neq \mathcal{F})$ satisfies the condition (E), by Corollary 4.4 above. A somewhat more elaborate version of the same idea enables us to drop the condition on the polynomials in order to get a more general result as follows (see [P1, Theorem 3.6]). Recall that if the Fréchet topology of A is given by a sequence (p_m), then each p_m is of type (E) if and only if A satisfies the condition (E) by p. 144 of (3); also, by [L2, Theorem 2], A satisfies the condition (E) if and only if A admits a *growth sequence*, i.e., there is a sequence (σ_n) of positive reals such that $\sigma_n \pi_n(x) \to 0$ for each $x \in A$ (from this condition, it is easy to see that $\mathcal{F}$ does not satisfy the condition (E)).

Theorem 5.5. *Let A be a FrAPS. Then $A \neq \mathcal{F}$ if and only if $A = \varprojlim A_m$, where each A_m is a BAPS. In particular, A satisfies the condition (E).* □

As discussed above, it is clear that every FrAPS satisfying the condition (E) has a unique Fréchet topology [L2, Corollary 2]. Since $\mathcal{F}$ does not satisfy the condition (E), there may be some other FrAPS not satisfying this condition; but, by Theorem 5.5, that possibility is ruled out. Hence, we have the following theorem (see [P1, Theorem 4.1]).

Theorem 5.6. *Let A be a FrAPS such that $A \neq \mathcal{F}$. Then a homomorphism $\theta : B \to A$ from a Fréchet algebra B into A is continuous provided that the range of θ is not one-dimensional.* □

Since $\mathcal{F}$ does not admit a growth sequence, the result of Johnson [J, Theorem 9.1] (which was proved for the Banach algebra case with an indication that some condition such as the existence of a growth sequence is required in the Fréchet case) is, here, established for Fréchet algebras in a more general form, provided that $A \neq \mathcal{F}$. Thus there can be no special automatic discontinuity result in this case. As a corollary, we have the following result in automatic continuity theory (see [P1, Corollary 4.2]). Scheinberg proved the continuity of automorphisms of $\mathcal{F}$.

Corollary 5.7. *Let A be a FrAPS. Then every automorphism of A is continuous. In particular, A has a unique Fréchet topology.* □

In relation to the still unsolved "Michael problem" (whether every character on a commutative Fréchet algebra need be continuous) [M, §12], we shall see that the following question from p. 135 of [P1] may have some interest in automatic continuity theory, as we shall see below.

Question. Is every (surjective) homomorphism $\theta : B \to \mathcal{F}$ from a non-Banach Fréchet algebra B continuous?

We remark that Dales and McClure established the existence of a commutative, unital Banach algebra A having a totally discontinuous higher point derivation $(d_n)_{n\geq 1}$ of infinite order at a character ((d_n) is a sequence of linear functionals on A at a character satisfying the (normalized) Leibniz identity), and which is the domain of a discontinuous homomorphism onto $\mathcal{F}$ [D, Theorem 5.5.19]. They also asked (somewhat casually) if *every* discontinuous homomorphism from a Banach algebra into $\mathcal{F}$ had to be surjective. The above question has an obvious relation with this problem. On the other hand, there is a discontinuous homomorphism between two commutative unital Fréchet algebras having certain properties [A2, Theorem 8]; but, Allan used a continuous homomorphism from A into $\mathcal{F}$ in the construction. My second goal was to answer the Dales-McClure problem from 1977 (or, the above question, for that matter). Meanwhile, I received the Commonwealth Academic Staff Fellowship for 6 months to avail at University of Leeds, Leeds, U.K. We (myself, Professors Garth Dales and Charles Read (U.K.)) decided to take up this problem for study. We are pleased to report here that the prime results are included in Oxford's Graduate Texts in Mathematics written by Allan (and edited by Dales)

[A3]. We now summarize important results in automatic continuity theory, obtained in [DPR]. We shall feel free to use the terminology, conventions and notations established there.

Let $k \in \mathbb{N}$ be fixed. We write $\mathcal{F}_k$ for the algebra $\mathbb{C}[[X_1, X_2, \ldots, X_k]]$ of all formal power series in k commuting indeterminates $X_1, X_2, \ldots, X_k$, with complex coefficients. A generic element of $\mathcal{F}_k$ is denoted by

$$\sum_{J\in\mathbb{Z}^{+k}} \lambda_J X^J = \sum\{\lambda_{(j_1, j_2, \ldots, j_k)} X_1^{j_1} X_2^{j_2} \cdots X_k^{j_k} : (j_1, j_2, \ldots, j_k) \in \mathbb{Z}^{+k}\}.$$

The algebra $\mathcal{F}_k$ is a Fréchet algebra when endowed with the weak topology τ_c defined by the coordinate projections

$$\pi_I : \sum_{J\in\mathbb{Z}^{+k}} \lambda_J X^J \mapsto \lambda_I, \ \mathcal{F}_k \to \mathbb{C},$$

for each $I \in \mathbb{Z}^{+k}$. A defining sequence of seminorms for $\mathcal{F}_k$ is $(p_m^{'})$, where

$$p_m^{'}(\sum_{J\in\mathbb{Z}^{+k}} \lambda_J X^J) = \sum_{|J|\le m} | \lambda_J | \ (m \in \mathbb{N}).$$

A *Fréchet algebra of power series in k variables* (shortly: FrAPS in $\mathcal{F}_k$) is a subalgebra A of $\mathcal{F}_k$ such that (A, τ) is a Fréchet algebra containing the indeterminates $X_1, X_2, \ldots, X_k$, and such that the inclusion map $(A, \tau) \hookrightarrow (\mathcal{F}_k, \tau_c)$ is continuous (equivalently, the projections π_I, $I \in \mathbb{Z}^{+k}$, are continuous linear functionals on A).

We shall also require in future theorems formal power series algebras over certain semigroups. For example, if $S = \mathbb{Z}^+$ or $S = (\mathbb{Z}^+)^k$, where $k \in \mathbb{N}$, then $\mathcal{F}_S = \mathcal{F}$ or $\mathcal{F}_S = \mathcal{F}_k$, respectively. Now let $S = (\mathbb{Z}^+)^{<\omega}$, the abelian semigroup of all $\mathbb{Z}^+$-valued sequences that are eventually 0. The corresponding formal power series algebra over $(\mathbb{Z}^+)^{<\omega}$ is denoted by $\mathcal{F}_\infty$ (other notation: $\mathbb{C}[[X_1, X_2, \ldots]]$). The generic element of $\mathcal{F}_\infty$ again has the form

$$\sum_{r\in\mathbb{Z}^{+k}} \lambda_r X^r = \sum\{\lambda_{(r_1, r_2, \ldots, r_k)} X_1^{r_1} X_2^{r_2} \cdots X_k^{r_k} : (r_1, r_2, \ldots, r_k) \in \mathbb{Z}^{+k}\},$$

but now there is no restriction on the value of $k \in \mathbb{N}$. The algebra $\mathcal{F}_\infty$ is a Fréchet algebra when endowed with the weak topology τ_c defined by the coordinate projections $\pi_r : \sum \lambda_r X^r \mapsto \lambda_r$, $\mathcal{F}_\infty \to \mathbb{C}$, for each $r \in (\mathbb{Z}^+)^{<\omega}$. A defining sequence of seminorms for $\mathcal{F}_\infty$ is $(p_m^{'})$, where

$$p_m^{'}(\sum \lambda_r X^r) = \sum\{| \lambda_r | : r \in (\mathbb{Z}^+)^m, | r | \le m\} \ (m \in \mathbb{Z}^+).$$

A *Fréchet algebra of power series in* $X_1, X_2, \ldots$ (shortly: FrAPS in $\mathcal{F}_\infty$) is a subalgebra A of $\mathcal{F}_\infty$ such that (A, τ) is a Fréchet algebra containing the indeterminates $X_1, X_2, \ldots$, and such that the inclusion map $(A, \tau) \hookrightarrow (\mathcal{F}_\infty, \tau_c)$ is continuous (equivalently, the projections π_r, $r \in (\mathbb{Z}^+)^{<\omega}$, are continuous linear functionals on A).

For $m \in \mathbb{N}$, set

$$\mathcal{U}_m = \{f = \sum \{\lambda_r X^r : r \in (\mathbb{Z}^+)^{<\omega}\} \in \mathcal{F}_\infty : q_m(f) := \sum |\lambda_r| m^{|r|} < \infty\},$$

and then set $\mathcal{U} = \bigcap\{\mathcal{U}_m : m \in \mathbb{N}\}$. It is clear that each $(\mathcal{U}_m, q_m)$ is a commutative, unital BAPS in $\mathcal{F}_\infty$. Thus $(\mathcal{U}, (q_m))$ is a commutative, unital FrAPS in $\mathcal{F}_\infty$. The algebra $\mathcal{U}$ contains each monomial X^r. The algebra $\mathcal{U}$ was first introduced by Clayton [Rocky Mountain J. Math. 5 (1975), 337-344]; it is studied further in [DE, DPR]. It is noted in [DE] that the map

$$\phi \mapsto (\phi(X_n) : n \in \mathbb{N}),\ M(\mathcal{U}) \to \ell^\infty,$$

is a continuous bijection, where $M(\mathcal{U})$ is the character space of $\mathcal{U}$. It can be said that $\mathcal{U}$ is the *algebra of all entire functions on* ℓ^∞.

We shall also require in future theorems semigroup algebras over a semigroup S. Specifically, we are interested in the semigroup algebra $\ell^1(S)$, where $S = (\mathbb{Z}^+)^{<\omega}$. It consists of all sums $f = \sum_{s \in S} \alpha_s \delta_s$, where $\alpha_s \in \mathbb{C}$ $(s \in S)$, such that $\sum_{s \in S} |\alpha_s| < \infty$. Of course, this algebra is a Banach algebra for the norm $\|\cdot\|_1$, specified by $\|f\|_1 = \sum_{s \in S} |\alpha_s|$ $(f = \sum_{s \in S} \alpha_s \delta_s \in \ell^1(S))$, and w.r.t. a unique product $*$ again specified by the condition that $\delta_s * \delta_t = \delta_{st}$ for all $s, t \in S$. Clearly, there is a continuous embedding of $\mathcal{U}$ into $\ell^1(S)$.

Set $E = \ell^1(\mathbb{Z}^+)$, the Banach space, and recall that, for each $n \in \mathbb{N}$, the Banach space $\ell^1((\mathbb{Z}^+)^n)$ can be identified as a Banach space with the n-fold projective tensor product $E_n := \widehat{\bigotimes}^n E = E \hat{\otimes}_\pi \cdots \hat{\otimes}_\pi E$. As in [D, Example 2.2.46 (ii)], we form the *projective tensor algebra* of E; this is $\widehat{\bigotimes} E = \{u = (u_n) = \sum_n u_n : u_n \in E_n\ (n \in \mathbb{N})\}$, with product denoted by $\otimes$, so that $(u_p) \otimes (v_q) = (\sum_{p+q=r} u_p \otimes v_q : r \in \mathbb{Z}^+)$; we obtain a non-commutative, unital Fréchet tensor algebra over E w.r.t. the coefficientwise convergence topology defined by an increasing sequence $(\|\cdot\|_m)$ of seminorms, where $\|\sum_n u_n\|_m = \sum_{n=0}^m \|u_n\|_1$ $(u = \sum_n u_n \in \widehat{\bigotimes} E)$. As in [D], we have the concept of a *symmetric element* and a *symmetrizing map* $\tilde{\sigma}$. The subspace of $\widehat{\bigotimes} E$ consisting of the symmetric elements is denoted by $\widehat{\bigvee} E$; it is the range of the map $\tilde{\sigma}$, and is itself an algebra w.r.t. the product $\vee$, where $(u_p) \vee (v_q) = (\sum_{p+q=r} \tilde{\sigma}(u_p \otimes v_q) : r \in \mathbb{Z}^+)$; we obtain a commutative, unital Fréchet symmetric algebra over E w.r.t. the same topology defined by an increasing sequence $(\|\cdot\|_m)$ of seminorms.

It was shown in [D, §5.5] how to construct continuous higher point derivation (d_n) of infinite order on $\ell^1(S)$, inducing a continuous homomorphism from $(\ell^1(S), \|\cdot\|)$ into $(\mathcal{F}, \tau_c)$. However, it was not clear then how to modify this argument to obtain a continuous *embedding* of $\ell^1(S)$ into $\mathcal{F}$; such an embedding is exhibited in the following theorem (see [DPR, Theorem 10.1]).

Theorem 5.8. *(i) There is a continuous embedding θ of $\ell^1(S)$ into $(\mathcal{F}, \tau_c)$ such that $\theta(X_1) = X$, and so $\ell^1(S)$ is (isometrically isomorphic to) a BAPS. (ii) The Fréchet algebra $\mathcal{U}$ is (isometrically isomorphic to) a FrAPS.* □

Thus there is a FrAPS which is a test case for the prestigious Michael's problem [DPR, Corollary 10.3]. Not only this, but we have shown the somewhat surprising fact that the 'much bigger' semigroup algebra $\ell^1(S_c)$, where S_c denotes the free semigroup on c generators, is also a BAPS [DPR, Theorem 10.5]. Then we answer the Dales-McClure problem in a stronger form in the following theorem (see [DPR, Theorem 11.2]).

Theorem 5.9. *Let A be an $(F)-$algebra, and let $(d_n)_{n\geq 0}$ be a non-degenerate discontinuous higher point derivation on A. Then the map $\theta\ :\ a \mapsto \sum_{n=0}^{\infty} d_n(a)X^n,\quad A \to \mathcal{F}$, is an epimorphism.* □

As corollaries, if A is an (F)-algebra of power series, then the character π_0 is continuous [DPR, Corollary 11.5]; further, every (F)-algebra of power series has a unique (F)-algebra topology [DPR, Corollary 11.7], extending Corollary 4.2 of [P1] by another method. We are pleased to report here that the time-honored definitions of Banach and Fréchet (and, more generally, $(F-)$) algebras of power series contain a redundant clause of the continuity of the inclusion map (an important clause among the three clauses in these definitions); see [DPR, Corollaries 11.3 and 11.4]. However, we cannot exclude this clause from the respective definitions of Banach, Fréchet and $(F-)$ algebras of power series in $\mathcal{F}_k$ due to the following theorem (see [DPR, Theorem 12.3]).

Theorem 5.10. *There exists a Banach algebra $(\ell^1(S),\ \|\cdot\|)$ such that $\mathbb{C}[X_1,\ X_2] \subset \ell^1(S) \subset \mathcal{F}_2$, but such that the embedding $(\ell^1(S),\ \|\cdot\|) \to (\mathcal{F}_2,\ \tau_c)$ is not continuous.* □

In fact, we have a considerably weaker form of Theorem 5.9 in the several-variable case as follows (see [DPR, Theorem 12.1]).

Theorem 5.11. *Let $k \in \mathbb{N}$, let A be an (F)-algebra, and let $\theta : A \to \mathcal{F}_k$ be a homomorphism such that $\theta(A)$ is dense in $(\mathcal{F}_k, \tau_c)$. Assume that its separating space $\mathcal{S}(\theta)$ has finite codimension in $\mathcal{F}_k$. Then θ is an epimorphism.* □

During my Leeds visit, Professor Garth Dales drew my attention to his question about the uniqueness of the (F)-algebra topology for (F)-algebras of power series in $\mathcal{F}_k$, which was unsolved since 1978. I settle this in the affirmative for FrAPS in $\mathcal{F}_k$ [P3]. The proof goes via first establishing the results of (3) and then establishing the results of [P1] in the several-variable case; however, this extension is highly non-trivial as we shall see below, e.g., the detail study of a class of Beurling-Fréchet algebras of semiweight type is required (many thanks to Professor Charles Read who brought a particular example of a FrAPS in $\mathcal{F}_k$ (which does not satisfy the condition (E)) to my attention). We shall now state only important theorems from [P3]. First, we completely characterize FrAPS in $\mathcal{F}_k$ in the following theorem (see [P3, Theorem 3.1]).

Theorem 5.12. *Let A be a commutative, unital Fréchet algebra. Suppose that there exists a fixed $k \in \mathbb{N}$ such that A contains a closed maximal ideal M such that: (i) $\bigcap_{n\geq 1} \overline{M^n} = \{0\}$;*

and (ii) dim$(\overline{M^n}/\overline{M^{n+1}}) = C_{n+k-1,n}$ *for all* n*. Then* A *is a FrAPS in* $\mathcal{F}_k$*. The converse holds if the polynomials in* $X_1, X_2, \ldots, X_k$ *are dense in* A*.* □

Next, we define *Beurling-Fréchet algebras* $\ell^1(\mathbb{Z}^{+k}, \Omega)$ *of semiweight type* as follows. A *semiweight function* on $\mathbb{Z}^{+k}$ is a function $\omega : \mathbb{Z}^{+k} \to \mathbb{R}$ such that

$$\omega(M+N) \leq \omega(M)\omega(N),\ \omega(0) = 1 \,\text{and}\, \omega(N) \geq 0\,(M, N \in \mathbb{Z}^{+k});$$

a semiweight function is a *weight function* if for all $N \in \mathbb{Z}^{+k}$, $\omega(N) > 0$. Also, we say that ω is a *proper semiweight* if $\omega(N_0) = 0$ for some $N_0 \in \mathbb{N}^k$. Let $k \in \mathbb{N}$, and let

$$\ell^1(\mathbb{Z}^{+k}, \Omega) := \{f = \sum_{J\in\mathbb{Z}^{+k}} \lambda_J X^J \in \mathcal{F}_k : \sum_{J\in\mathbb{Z}^{+k}} \mid \lambda_J \mid \ \omega_m(J) < \infty \,\text{for all}\, m\},$$

where $\Omega = (\omega_m)$ is a separating and increasing sequence of semiweight functions on $\mathbb{Z}^{+k}$ defined by $\omega_m(J) = p_m(X^J)$. We call such a Beurling-Fréchet algebra $\ell^1(\mathbb{Z}^{+k}, \Omega)$ an *algebra of semiweight type.* For examples of these algebras, we refer to [P3]; in particular, for $k = 2$, $\ell^1(\mathbb{Z}^{+k}, \Omega)$ is either $\mathcal{F}_2$ or A_X (Read's example) or A_Y (all these algebras are local, that is, having the unique maximal ideal). For the five important properties of these algebras, we refer to [P3, pp. 38-39]. A Fréchet algebra $(B, (p_m))$ is said to be a *Fréchet algebra with p.s.g.s* $x_1, x_2, \ldots, x_k$ if each $y \in B$ is of the form

$$y = \sum_{J\in\mathbb{Z}^{+k}} \lambda_J\, x^J = \sum\{\lambda_{(j_1, j_2, \ldots, j_k)}\, x_1^{j_1} x_2^{j_2} \cdots x_k^{j_k} \ : \ (j_1, j_2, \ldots, j_k) \in \mathbb{Z}^{+k}\},$$

for λ_J complex scalars such that $\sum_{J\in\mathbb{Z}^{+k}} \mid \lambda_J \mid \ p_m(x^J) < \infty$ for all m. We have several-variable analogues of Theorems 4.1 and 5.2 as follows.

Theorem 5.13. *Let* A *be a FrAPS in* $\mathcal{F}_k$*. Suppose that* $X_1, X_2, \ldots, X_k$ *are p.s.g.s for* A*. Then* A *is the Beurling-Fréchet algebra* $\ell^1(\mathbb{Z}^{+k}, \Omega)$ *for an increasing sequence* Ω *of (semi)weight functions on* $\mathbb{Z}^{+k}$*.* □

Theorem 5.14. *Let* A *be a FrAPS in* $\mathcal{F}_k$*. Then* A *is either a Beurling-Fréchet algebra* $\ell^1(\mathbb{Z}^{+k}, \Omega)$ *of semiweight type or the Fréchet topology* τ *of* A *is defined by a sequence* (p_m) *of norms.* □

As corollaries, we have the following curious characterizations of a Beurling-Fréchet algebra $\ell^1(\mathbb{Z}^{+k}, \Omega)$ of semiweight type as a Fréchet algebra (cf. Corollaries 5.3 and 5.4).

Corollary 5.15. *Let* A *be a FrAPS in* $\mathcal{F}_k$*. Then* A *is equal to a Beurling-Fréchet algebra* $\ell^1(\mathbb{Z}^{+k}, \Omega)$ *of semiweight type if and only if the Fréchet topology of* A *is defined by a sequence* (p_m) *of proper seminorms. In particular,* $A = \mathcal{F}_k$ *if and only if the Fréchet topology of* A *is defined by a sequence* (p_m) *of proper seminorms with finite-codimensional kernels.* □

In fact, we have an Arens-Michael representation of A as follows (cf. Corollary 5.4).

Corollary 5.16. *Let A be a FrAPS in $\mathcal{F}_k$ such that the polynomials are dense in A. Then $A \neq \ell^1(\mathbb{Z}^{+k}, \Omega)$ the Beurling-Fréchet algebra of semiweight type, if and only if $A = \varprojlim A_m$, where each A_m is a BAPS in $\mathcal{F}_k$.* □

We have the several-variable analogues of Theorems 5.5 and 5.6 as follows; for some important remarks on Theorem 5.17, we refer to [P3, p. 43].

Theorem 5.17. *Let A be a FrAPS in $\mathcal{F}_k$. Then $A \neq \ell^1(\mathbb{Z}^{+k}, \Omega)$ the Beurling-Fréchet algebra of semiweight type, if and only if $A = \varprojlim A_m$, where each A_m is a BAPS in $\mathcal{F}_k$. In particular, A satisfies the condition (E).* □

Theorem 5.18. *Let A be a FrAPS in $\mathcal{F}_k$ such that $A \neq \ell^1(\mathbb{Z}^{+k}, \Omega)$ the Beurling-Fréchet algebra of semiweight type. Then a homomorphism $\theta : B \to A$ from a Fréchet algebra B is continuous provided that the range of θ is not one-dimensional.* □

Now, we have the following theorem, whose proof is given by another, short and elegant method (see [P3, Theorem 4.2]).

Theorem 5.19. *Let A be a Beurling-Fréchet algebra $\ell^1(\mathbb{Z}^{+k}, \Omega)$ of semiweight type. Then A has a unique Fréchet topology.* □

As a corollary of the above two theorems, we have the following result, answering a query from [DPR, p. 134] for FrAPS in $\mathcal{F}_k$ (see [P3, Corollary 4.2]).

Corollary 5.20. *Every FrAPS A in $\mathcal{F}_k$ has a unique Fréchet topology.* □

We now list the following questions, which may have some interest in automatic continuity theory (see [P3, p. 45]).

Question 1. Let A be a Beurling-Fréchet algebra $\ell^1(\mathbb{Z}^{+k}, \Omega)$ of semiweight type. Is every automorphism of A continuous?

We conjecture that Question 1 may have an affirmative answer. More generally, we have the following

Question 2. Is every homomorphism $\theta : B \to \mathcal{F}_k$ $(k > 1)$ from a Fréchet algebra B continuous?

The above question was partially answered in [DPR]; see Theorems 5.10 and 5.11. We use the structure of the closed ideals and their powers to establish the uniqueness of the Fréchet topology for FrAPS in $\mathcal{F}_k$; this is not known for the larger algebra $\mathcal{F}_\infty = \mathbb{C}[[X_1, X_2, \ldots]]$ [R]. We cannot apply our approach to obtain such result for FrAPS in $\mathcal{F}_\infty$. So it is of interest to know whether every FrAPS in $\mathcal{F}_\infty$ (except $\mathcal{F}_\infty$ itself) has a unique Fréchet topology. In other words, we have the following natural question.

Question 3. Is there any other proper, unital subalgebra of $\mathcal{F}_\infty$, with two inequivalent Fréchet topologies? In particular, is there any other proper subalgebra of $\mathcal{F}_\infty$ which is closed under the topology imposed by Read on $\mathcal{F}_\infty$ and which is also FrAPS in $\mathcal{F}_\infty$ in its "usual" topology?

To answer the latter part of the above question, the "natural" extension, $\ell^1((\mathbb{Z}^+)^{<\omega}, \Omega)$ of Beurling-Fréchet algebras of semiweight type, would be an easy target. Not only this, but we shall see that an attempt to answer this question for the test case $\mathcal{U}$ for the prestigious Michael problem would lead us to an affirmative answer to this problem! Also, we have the following curious question, whose solution we have given in [P5].

Question 4. Does there exist a Fréchet algebra with infinitely many inequivalent Fréchet topologies?

We first give the following result by noticing that a FrAPS A in $\mathcal{F}_\infty$ satisfies the condition (E) if there is a sequence $(\gamma_K : K \in \mathbb{N}^k, k \in \mathbb{N})$ of positive reals such that $(\gamma_K^{-1}\pi_K)$ is equicontinuous (see [P3, Theorem 4.4]).

Theorem 5.21. *Let A be a FrAPS in $\mathcal{F}_\infty$ satisfying the condition (E), above, and let $\phi : B \to A$ be a homomorphism from a Fréchet algebra B into A such that $X_1 \in \phi(B)$. Then ϕ is continuous. In particular, every automorphism of A is continuous, and A has a unique Fréchet topology.* □

We remark that the Fréchet topology of A in the above theorem is defined by a sequence of norms. We have the following "natural" generalization of Theorem 5.8 (see [P3, Theorem 4.5]).

Theorem 5.22. *Let A be a FrAPS in $\mathcal{F}_\infty$, with its topology defined by a sequence (p_m) of norms. Suppose that A is a graded subalgebra of $\mathcal{F}_\infty$. Then there is a continuous embedding θ of A into $(\mathcal{F}, \tau_c)$ such that $\theta(X_1) = X$, and so A is (isometrically isomorphic to) a FrAPS in $\mathcal{F}$. In particular, A has a unique Fréchet topology.* □

We now turn our attention to develop methods to give two (resp., countably many) inequivalent Fréchet topologies to certain Fréchet algebras (to answer Question 4). There are two natural ways for this: (A) Loy gave a simple method of using discontinuity of (point) derivations (of finite order) for constructing commutative Banach algebras with non-unique complete norm topology [L3]. We exploit the idea herein for the Fréchet case; (B) Read developed the "tensor product by rows" method to give another inequivalent Fréchet topology τ_0 on $\mathcal{F}_\infty$, using the discontinuous linear functional. We exploit his method to generate countably many mutually inequivalent Fréchet topologies on $\mathcal{F}_\infty$ and $\mathcal{F}_\infty \oplus \mathcal{F}_\infty$ (we remark that such examples are not even known in the Banach case so far).

Let A be a commutative Fréchet algebra and let M be a Fréchet space which is a commutative A-module. For such A and M, $H^1(A, M)$ (resp., $H^1_C(A, M)$) is the space of (continuous) derivations of A into M (if $M = \mathbb{C}$, then it is the space of point derivations at some character ϕ). Our first result is the following theorem (see [P4, Theorem 2.1]).

Theorem 5.23. *Let $D : A \to M$ be a non-zero derivation from a commutative Fréchet algebra A into a commutative Fréchet A-module M vanishing on a dense subset of A. Then the algebra $\overline{A_D}$ admits two inequivalent Fréchet topologies.* □

As a corollary, we have the following special case (see [P4, Corollary 2.2]). We define the seminorms q_k and $q_{k,D}$ for a more general case, below.

Corollary 5.24. *Let A be the Fréchet algebra $(\mathcal{F}_\infty, \tau_0)$ and let D be the derivation $\partial/\partial X_0$. Then $\overline{(\mathcal{F}_\infty)_D}$ admits another Fréchet topology τ_D, generated by $(q_{k,D})$, different from $\tau_0 + \tau_0$, generated by (q_k).* □

Although the Fréchet topology $\tau_0 + \tau_0$ of $\mathcal{F}_\infty \oplus \mathcal{F}_\infty$ is not obtainable by our approach, other inequivalent Fréchet topologies on $\mathcal{F}_\infty \oplus \mathcal{F}_\infty$ may be constructed, as we shall see below. We have, in fact, a more general result than Theorem 5.23 as follows (see [P4, Theorem 2.3]). Set $\mathcal{U} = A \oplus M$, where $(a, x)(b, y) = (ab, a \cdot y + b \cdot x)$ for $a, b \in A$ and $x, y \in M$. Then $\mathcal{U}$ is a commutative algebra with $\operatorname{Rad} \mathcal{U} = \operatorname{Rad} A \oplus M$. Let $D : A \to M$ be a derivation, and set

$$q_k((a, x)) = p_k(a) + p_k(x),\ q_{k,D}((a, x)) = p_k(a) + p_k(D(a) - x)\ (a \in A,\ x \in M).$$

Theorem 5.25. *The algebra $\mathcal{U}$ is a Fréchet algebra with respect to both (q_k) and $(q_{k,D})$. The two topologies are equivalent if and only if D is continuous.* □

In the above situations the ideal adjoined was always nilpotent of index two; we now consider how to obtain more general ideals in the following theorems (see [P4, Theorems 3.4, 4.1 and 4.2]).

Theorem 5.26. *Let A be a commutative Fréchet algebra, $D = \{D_1, \ldots, D_r\}$ a higher point derivation of rank r such that D is a set of discontinuous functionals. Then $\overline{A_D}$ admits two inequivalent Fréchet topologies and has nilpotent elements of index r.* □

Loy raised the following question in [L3]: whether quasinilpotent non-nilpotent radicals are obtainable using the totally discontinuous higher point derivation of infinite order on a commutative Banach algebra. We affirmatively answer this question in the Fréchet case (this is not possible in the Banach case due to Theorem 5.9, exhibiting a significant difference between Banach algebras and Fréchet algebras). We refer to [D, 2.2.46 (ii)] for more details on the Dales-McClure Banach algebra $\widehat{\bigvee}_\omega E$ (e.g., when $E = \ell^1(\mathbb{Z}^+)$, it is a weighted Banach subalgebra of a commutative, unital Fréchet symmetric algebra $\widehat{\bigvee} E$, discussed preceding to Theorem 5.8), and to know about the two inequivalent Fréchet topologies on $\overline{A_D}$, follow either Theorems

5.25 or 5.26. We note that the Dales-McClure Fréchet algebra can be constructed along the lines of the Dales-McClure Banach algebra by replacing a weight ω on $\mathbb{Z}^+$ by an increasing sequence $W = (\omega_k)$ of weights on $\mathbb{Z}^+$. Thus, we have $\widehat{\bigvee}_W E = \bigcap_{k=1}^{\infty} \widehat{\bigvee}_{\omega_k} E$, an analogue of the Beurling-Fréchet algebra (see Example 1.2 of (3)). We remark that the test case $\mathcal{U}$ for Michael's acclaimed problem is a weighted Fréchet symmetric algebra over the Banach space $E = \ell^1(\mathbb{Z}^+)$, and thus, we are pleased to report here that this paper turns out to be the root of the ideas to affirmatively solve this problem (see [P7]).

Theorem 5.27. *Let A be the Dales-McClure Banach (resp., Fréchet) algebra, $D = (D_i)$ a totally discontinuous higher point derivation of infinite order at a (continuous) character ϕ. Then the Fréchet algebra $\overline{A_D}$ admits two inequivalent Fréchet topologies and has quasinilpotent non-nilpotent elements.* □

In [R], Read showed that the derivation $\partial/\partial X_0$ is discontinuous on $\mathcal{F}_\infty = \mathbb{C}[[X_0, X_1, \dots]]$ w.r.t. the Fréchet topology τ_0 and that X_0 lies in the closure of the coefficient algebra $\mathcal{A}_0 = \mathbb{C}[[X_1, X_2, \dots]]$ (i.e., $X_n \to X_0$ as $n \to \infty$ w.r.t. τ_0). So, one has $X_n - X_0 \to 0$ yet $\partial/\partial X_0(X_0 - X_n) = 1$. Thus 1 is in the separating subspace for $\partial/\partial X_0$; since the subspace is an ideal, it is the whole algebra. Thus $\mathrm{Im}(\partial/\partial X_0) = \mathcal{F}_\infty$, establishing the Singer-Wermer conjecture (whether the image of a derivation map (continuous or not) on a Banach algebra is contained in the radical) in the negative in the Fréchet case. Thomas (U.S.A.) already showed that the image of a discontinuous derivation on a commutative Banach algebra is contained in the radical [T]. Thus the situation on Fréchet algebras is markedly different from that on Banach algebras. In fact, it is surprising to observe that for each $i \in \mathbb{N}$, the natural derivation $\partial/\partial X_i$ is discontinuous w.r.t. the Fréchet topology τ_i (defined below), and following the same arguments above, $\mathrm{Im}(\partial/\partial X_i) = \mathcal{F}_\infty$. This shows that for each $i \in \mathbb{N}$, $\partial/\partial X_i$ vanishes on a dense subset $\mathcal{A}_i$ of $(\mathcal{F}_\infty, \tau_i)$. Not only this, but it is easy to see that for $i \neq j$, the Fréchet topologies τ_i and τ_j on $\mathcal{F}_\infty$ are mutually inequivalent, because the identity map is continuous in neither direction as $X_n \to X_i$ in $(\mathcal{F}_\infty, \tau_i)$ whereas $X_n \to X_j$ in $(\mathcal{F}_\infty, \tau_j)$. Further, for $i \neq j$, $\partial/\partial X_i$ are continuous on $(\mathcal{F}_\infty, \tau_j)$ as $X_n \to X_j$ and $\partial/\partial X_i(X_j - X_n) = 0$. We remark that it is not so straightforward to show that $(\mathcal{F}_\infty, \tau_i)$ is a Fréchet algebra for each $i \in \mathbb{N}$ in the following theorem [P5, Theorem 2.1]; one requires to carefully apply the Read's method for $i \in \mathbb{N}$ (the case $i = 0$ gives the topology τ_0).

Theorem 5.28. *Let $i \in \mathbb{Z}^+$ be fixed. $(\mathcal{F}_\infty, \tau_i)$ is complete w.r.t. the seminorms $(p'_{m,i})$. The derivation $\partial/\partial X_i : (\mathcal{F}_\infty, \tau_i) \to (\mathcal{F}_\infty, \tau_i)$ is discontinuous, and its separating subspace is all of $\mathcal{F}_\infty$.* □

Now, combining the method discussed in (A) [P4] and the fact that $\mathcal{F}_\infty$ admits countably many mutually inequivalent Fréchet topologies as above, we are in position to do miracle on

$\mathcal{F}_\infty \oplus \mathcal{F}_\infty$. We now give families of (in)equivalent Fréchet topologies on this algebra. First, let $\mathcal{F}_\infty$ be a Fréchet algebra w.r.t. τ_i, generated by $(p'_{m,i})$ for each $i \in \mathbb{Z}^+$, and w.r.t. τ_c, generated by (p_m). Then, on $\mathcal{F}_\infty \oplus \mathcal{F}_\infty$, we give several Fréchet topologies as follows: (a) $\tau_c + \tau_c$, generated by (q_m), where $q_m = p_m + p_m$, (b) for each $i \in \mathbb{Z}^+$, $\tau_i + \tau_i$ generated by $(q'_{m,i})$, where $q'_{m,i} = p'_{m,i} + p'_{m,i}$ (see preceding to Theorem 5.25), (c) for each $i \in \mathbb{Z}^+$, τ_{∂_i} and τ'_{∂_i} generated by (q_{m,∂_i}) and (q'_{m,∂_i}), respectively, and induced by ∂_i (see preceding to Theorem 5.25). We briefly see whether they are (in)equivalent to each other; we use "=" for equivalency of topologies. First, $\mathcal{F}_\infty$ has been completed by the adjunction of a radical so that $\mathcal{F}_\infty$ is dense in $(\mathcal{F}_\infty \oplus \mathcal{F}_\infty, \tau_i + \tau_i)$ (for this, recall that, for each $i \in \mathbb{Z}^+$, $\partial_i = \partial/\partial X_i$ vanishes on a dense subset $\mathcal{A}_i$ due to the unique property of τ_i as above). In fact, we have a more stronger result as follows. For each $i \in \mathbb{Z}^+$, ∂_i on $\mathcal{F}_\infty$ induces a natural derivation $D_i : (a, x) \mapsto (0, \partial_i(a))$ on $\mathcal{F}_\infty \oplus \mathcal{F}_\infty$. Then, for each $i \in \mathbb{Z}^+$, $\tau_{\partial_i} = \tau_c + \tau_c$ (resp., $\tau'_{\partial_i} \neq \tau_i + \tau_i$) since ∂_i is (dis)continuous on $(\mathcal{F}_\infty, \tau_c)$ (resp., $(\mathcal{F}_\infty, \tau_i)$) if and only if D_i is (dis)continuous on $(\mathcal{F}_\infty \oplus \mathcal{F}_\infty, \tau_c + \tau_c)$ (resp., $(\mathcal{F}_\infty \oplus \mathcal{F}_\infty, \tau_i + \tau_i)$) by Theorem 5.25. Also, for each $i \in \mathbb{Z}^+$, $\tau_i + \tau_i \neq \tau_c + \tau_c$; in fact, $\tau_i + \tau_i \neq \tau_j + \tau_j$ for $i \neq j$. Similarly, for each $i \in \mathbb{Z}^+$, $\tau'_{\partial_i} \neq \tau_c + \tau_c$. Now, for $i \neq j$, $\tau_{\partial_i} = \tau_{\partial_j}$ (resp., $\tau'_{\partial_i} \neq \tau'_{\partial_j}$) if and only if D_i is (dis)continuous on $(\mathcal{F}_\infty \oplus \mathcal{F}_\infty, \tau_{\partial_j})$ (resp., $(\mathcal{F}_\infty \oplus \mathcal{F}_\infty, \tau'_{\partial_j})$) and D_j is continuous on $(\mathcal{F}_\infty \oplus \mathcal{F}_\infty, \tau_{\partial_i})$. Interestingly, the Singer-Wermer conjecture holds for D_i on $(\mathcal{F}_\infty \oplus \mathcal{F}_\infty, \tau_{\partial_i})$ as $\mathrm{Im} D_i = 0 \oplus \mathcal{F}_\infty \subset \mathrm{Rad}(\mathcal{F}_\infty \oplus \mathcal{F}_\infty) = \mathcal{F}^\bullet_\infty \oplus \mathcal{F}_\infty$.

After having worked on several long standing problems in automatic continuity theory (including the (non)-uniqueness of the Fréchet topology on certain Fréchet algebras), we now have enough ammunition to solve the still unsolved, prestigious Michael problem in Fréchet algebra theory. For brevity, we shall only sketch the two approaches here (see [P7]). Our first approach is to show that the test case, the Fréchet algebra $\mathcal{U}$, for this problem is, in fact, $\mathcal{F}$, if there exists a discontinuous character ϕ on $\mathcal{U}$. The starting point of this approach is that $\mathcal{U}$ can be viewed as the weighted Fréchet symmetric algebra $\widehat{\bigvee}_W E$, where $W = (\omega_m)$, where $\omega_m(|r|) = m^{|r|}$ $(r \in (\mathbb{Z}^+)^{<\omega})$, is an increasing sequence of weights on $\mathbb{Z}^+$ and $E = \ell^1(\mathbb{Z}^+)$ [P4]. Now, the Dales-McClure method allows us to generate a non-degenerate, totally discontinuous higher point derivation (d_n) on $\mathcal{U}$ at ϕ. Then, by Theorem 5.9, the homomorphism θ from $\mathcal{U}$ into $\mathcal{F}$ is, in fact, surjective. Also, by Theorem 5.8 (ii), $\mathcal{U}$ is a FrAPS. So, we have $\mathcal{U} = \mathcal{F}$, a contradiction to the fact that $\mathcal{U}$ is semisimple and $\mathcal{F}$ is local.

Our second approach rests on applying the Read's method from [R] to construct another Fréchet topology on $\mathcal{U}$, inequivalent to the "usual" Fréchet topology, generated by (q_m). Again, the existence of a discontinuous character on $\mathcal{U}$ allows us to consider a discontinuous linear functional on ℓ^1, and further on the Fréchet space $\mathcal{U}^{(1)}$ the closed linear subspace of $\mathcal{U}$, spanned by X_i $(i \in \mathbb{N})$ and induced by $\ell^1 = \mathcal{U}^{(1)}_1$ the closed linear subspace of a Banach algebra

$(\mathcal{U}_1, q_1)$, spanned by X_i $(i \in \mathbb{N})$. Now, we consider the sequence $(\phi_n)_{n\geq 0}$ of linear functionals on $\mathcal{U}^{(1)}$ such that ϕ_0 is the discontinuous linear functional on $\mathcal{U}^{(1)}$ and other ϕ_n are the "weighted" coordinate functionals on $\mathcal{U}^{(1)}$. This sequence generates the sequence of seminorms on $\mathcal{U}$ (here, one requires to use the notion of "tensor product by rows" [R]), which defines another Fréchet topology on $\mathcal{U}$, a contradiction to the fact that $\mathcal{U}$ has a unique Fréchet topology, since $\mathcal{U}$ is a semisimple FrAPS (in $\mathcal{F}_\infty$) [C, P1, P3, DPR]. There are several important implications of this gigantic result (e.g., every non-commutative Fréchet algebra is functionally continuous; every non-commutative, semisimple Fréchet algebra B has a unique Fréchet topology and every derivation on B is automatically continuous) in automatic continuity theory. Since this paper is under preparation, interested readers are suggested to contact author to know more about these results. Thus, we have the following

Theorem 5.29. *Every character on a Fréchet algebra (commutative or not) is automatically continuous.* □

An important subject in the Fréchet algebra theory is the question of the existence of analytic structure in spectra. In [P2, P6], we are specifically concerned with the determination of sufficient conditions for the existence of local analytic structure in the spectrum of a Fréchet algebra by studying the ideal structure of the algebra (see Main Theorems). As consequences, we characterize locally Riemann (resp., Stein) algebras (new notions) by intrinsic properties within Fréchet algebras. We recall that the spectrum $M(A)$ (with the Gel'fand topology) has an analytic variety at $\phi \in M(A)$ if there is a subvariety D containing 0 of a domain in some $\mathbb{C}^k$ and a continuous injection $f : D \to M(A)$ such that $f(0) = \phi$ and $\hat{x} \circ f \in \mathrm{Hol}(D)$ for all $x \in A$ (if $k = 1$, D is the open unit disc). We call a Fréchet algebra A a locally Riemann (resp., Stein) algebra if a non-empty part of $M(A)$ can be given the structure of a Riemann surface (resp., (reduced) Stein space) in such a way that the completion in the compact open topology of the algebra of Gelfand transforms of elements of A, restricted to this part, is the Fréchet algebra of all holomorphic functions on this Riemann surface (resp., Stein space). In [P6], we establish the Gleason result for finitely generated ideals in the context of Fréchet algebras, providing an affirmative answer to a question about the Gleason result in Fréchet algebras, posed by Carpenter for uniform Fréchet algebras in 1970. As a welcome bonus, *locally Stein algebras* are completely characterized, and, as an application of this characterization, an affirmative answer to the Gleason problem (1964) for such algebras is provided through the functional analytic approach, recapturing all the classical results on the Gleason problem in the theory of SCV.

6. Our Research Papers

(1) S. J. Bhatt and S. R. Patel, *A note on Banach algebras with a power series generator*, "Prajna" - Jr. Pure and Applied Sciences 11 (2001), 32-37.

(2) S. J. Bhatt, H. V. Dedania and S. R. Patel, *Fréchet algebras with a Laurent series generator and annulus algebras*, Bull. Austral. Math. Soc. 65 (2002), 371-383.
(3) S. J. Bhatt and S. R. Patel, *On Fréchet algebras of power series*, Bull. Austral. Math. Soc. 66 (2002), 135-148.

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[P2] S. R. Patel, *Fréchet algebras, formal power series, and analytic structure*, J. Math. Anal. Appl. 394 (2012), 468-474.

[P3] S. R. Patel, *Uniqueness of the Fréchet algebra topology on certain Fréchet algebras*, Studia Math. 234 (2016), 31-47.

[P4] S. R. Patel, *On discontinuity of derivations, inducing non-unique complete metric topologies*, pre-print (refereed), pp. 1-26.

[P5] S. R. Patel, *On Singer-Wermer conjecture in Fréchet algebras*, pre-print (refereed), pp. 1-26.

[P6] S. R. Patel, *On functional analytic approach for the famous Gleason problem in the theory of SCV*, pre-print (refereed), pp. 1-39.

[P7] S. R. Patel, *On an affiramtive solution to Michael's acclaimed problem in the theory of Fréchet algebras, with applications to automatic continuity theory*, pre-print, pp. 1-81.

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