Mathematics Today Volume 36 Issue 1 (September 2020), 96-101
ISSN 0976-3228, e-ISSN 2455-9601

# FRACTIONAL BROWNIAN MOTION AND PREDICTABILITY INDEX IN FINANCIAL MARKET

V R SHAH

## 1. Tribute

My association with Bhatt Sir was a point of inflection in my life. It's an evening of one day, end of June, in 1993. Bhatt Sir came to see a student of his village at B.Ed hostel, behind Nehru Hall. During his visit to the hostel he found other two boys Mukesh Shimpi and myself. We had just completed our M. Sc. study in Mathematics from the Department of Mathematics, Sardar Patel University where we learned a subject under Prof Bhatt Sir. So he knew us. Moreover, we both obtained good results and waere considered to be good students having inclination towards the subject. He asked us what we were doing there? We said, Sir we are pursuing B. Ed. He asked us why? We said: Sir our parent's financial conditions would not allow us to stretch our study and we need to have a job at the earliest so that we can contribute to our respective families. Also, after studying heavy stuff of pure mathematics at M.Sc. we were enjoying B.Ed. He immediately asked both of us to come to his home. We went to his home and after almost 4 hours of discussion he convinced us to leave B.Ed., start M.Phil., and leave rest to God. We both got the job just after six months, myself at V P Science College, where again my destiny made me to meet the fatherly Head Late Prof. B. D. Patel, a true admirer of Bhatt Sir, and Shimpi at B V M Engineering college. Since then our associations with Bhatt Sir and his family are like family members. Bhatt Sir and Dedania Sir were having perfect tuning at all levels so that helped me a lot when I was doing Ph.D. under the guidance of Prof. H. V. Dedania. Bhatt Sir helped me a lot during my Ph.D. and his clicking on some ideas were sharp and pin pointed. Recently in a talk about numerous applications of mathematics in engineering, he suggested that "Yes that's the work of mathematicians and engineers to explore but during the process many engineers too contributed a lot to mathematics". Do search and even you can write a good article on it. His latest thoughts, reading were on "The Unity in Sciences". Can the various natural sciences be unified into a single science, and can theories within a single science (e.g., general theory of relativity and quantum theory in physics, or models of evolution and development in biology) be unified into one theory? I still remember his words while I asked a question when we were walking at Shastri Ground, Vallabh Vidyanagar. I asked: Sir

2010 *Mathematics Subject Classification.* 46J05, 22B10.

*Key words and phrases.* Brownian motion, Fractal Brownian motion, volatility, Hurst exponent, Fractal dimension, predictability index.

it's more than 10 years of retirement and you are still much active in doing mathematics. Do you not feel exhausted? His reply was...“My aim is to receive Field Medal, if not this in life then in next life”. Besides a great mathematician Bhatt Sir was a great human being. My obeisance to Bhatt Sir who has achieved the state of mathematical transcendence.

## 2. Our Joint Research

My collaborative research with Bhatt Sir is in two papers jointly with Prof. Haresh Dedania. In the first paper we have introduced predictability index using fractal dimension while in the second paper we did fractal dimensional analysis in financial time series. I have also presented the work at an international conference held at Hong Kong in December 2013.

Brownian motion was introduced by Louis Bachelier in 1900 in his thesis “The theory of speculation” on the mathematical modeling of the Paris stock exchange. However, it was only in 1905 that Albert Einstein could sufficiently explain Brownian motion using a probabilistic model. The majority of approaches towards analysis in mathematical finance deals with Brownian motion as a source of randomness.

**Definition 1.** [1] A discrete Brownian motion *(BM) is a real valued stochastic process* $t_i \to B(t_i)$ *over discrete time values* $t_i$, $i = 0, 1, 2, \ldots$ *such that*

(1) $B(t_0) = B(0) = 0$;

(2) *the increments* $\Delta B(t_j, t_i)$ *and* $\Delta B(t_l, t_k)$ *for* $(0 \leq i < j \leq k < l)$ *are independent random variables, where* $\Delta B(t_n, t_m) = B(t_n) - B(t_m)$;

(3) *the increments* $\Delta B(t_j, t_i)$ *are normal random variables with mean* $E\left(\Delta B(t_j, t_i)\right) = 0$ *and variance* $Var\left(\Delta B(t_j, t_i)\right) = t_j - t_i$ $(j > i)$.

Taking BM as the source of randomness, the celebrated Black-Schole-Mortion theory (1973) develops option pricing formula involving stochastic calculus which is based on integration with respect to the Brownian motion under the assumption that the market follows a Brownian path. In 1968 Mandelbrot and van Ness suggested the use of fractional Brownian motion as a source of randomness in financial market in their seminal paper [3]. Also in 1988 Lo and MacKinley infer that the processes of observable market values seem to exhibit serial correlation, there by suggesting the use of fBM.

**Definition 2.** [1] A discrete fractional Brownian motion *(fBM) is a real valued stochastic process* $t_i \to B_H(t_i)$ *over discrete time values* $t_i, i = 0, 1, 2, \ldots$ *such that*

(1) $B_H(t_0) = B_H(0) = 0$;

(2) *The increments* $\Delta B_H(t_j, t_i)$ *are normal random variable with the mean* $E\left(\Delta B_H(t_j, t_i)\right) = 0$ *and variance* $Var\left(\Delta B_H(t_j, t_i)\right) = (t_j - t_1)^{2H}$ $(j > i)$ *where* $H$ *is the Hurst exponent satisfying* $0 < H < 1$.

When $H = 0.5$ it becomes a BM. Thus, fBM includes BM.

**Theorem 2.1.** [8] *The process $t_i \to B_H(t_i)$ is known to satisfy the following: For all $i$ and $j$,*

(1) $E(B_H(t_i)) = 0$.
(2) $E\left(B_H(t_i)B_H(t_j)\right) = \frac{1}{2}\left[|t_i|^{2H} + |t_j|^{2H} - |t_i - t_j|^{2H}\right]$.
(3) *The increments $\Delta B_H(t_i, t_j) = B_H(t_i) - B_H(t_j)$ satisfy for all $i$ and $j$,*
   (a) $E(\Delta B_H(t_i, t_j)) = 0$;
   (b) $E\left((\Delta B_H(t_i, t_j))^2\right) = |t_i - t_j|^{2H}$.
(4) *The covariance of increments $\Delta B_H(t_i, t_j)$ and $\Delta B_H(t_j, t_k)$ is*

$$E(\Delta B_H(t_i, t_j)\Delta B_H(t_j, t_k)) = \frac{1}{2}\left[t_i^{2H} - t_j^{2H} - (t_i - t_j)^{2H}\right].$$

*So except for $H = \frac{1}{2}$, the increments of fBM are not independent.*
(5) *A fBM is known to exhibit self similarity; the resulting curve is a fractal whose fractal dimension is given by $D = 2 - H$.*

**An Application of Hurst Exponent in Finance**

The Hurst exponent $H$ is used as a measure of long term memory of time series. The concept was used by Harold Edwin Hurst (1880-1978) while determining optimum dam sizing for the Nile river's volatile rain and drought conditions that had been observed over a long period of time. Many years later, while investigating the fractal nature of financial markets  specifically, the tendency of a time series to regress strongly to its mean or to cluster in a direction. A noted mathematician Benoit Mandelbrot happened to stumble across Hursts work. By recognizing the potential of his work, Mandelbrot introduced the concept of fractal geometry in Hursts honor. It is also called the Generalized Hurst Exponent. Meaningful values of $H$ are in the range of (0,1). There are many techniques for estimating the Hurst exponent. We shall use the classical "Rescaled Range" $\frac{R}{S}$ analysis. It provides a direct estimation of the Hurst exponent which is a meaningful indicator of the state of randomness of a time series. It is observed by many researchers that an estimation of the Hurst exponent may yield some valuable information on the long term behavior of a particular asset.

**Definition 3.** [5] *The Hurst exponent, $H$ is defined, in terms of the asymptotic behavior of the rescaled range of the time span of a time series as*

$$\lim_{n\to\infty} E\left[\frac{R(n)}{S(n)}\right] = Cn^H,$$

*where $R(n)$ is the range of the first $n$ values, $S(n)$ is the standard deviation of the first $n$ values, $E(x)$ is the expected value, $n$ is the number of data points in a time series, and $C$ is a constant.*

Using the Hurst exponent we can classify time series into types and gain some insight into their dynamics [4, 5]. Here are some types of time series and the fractal dimension associated with each of them.

(1) **A Brownian time series**:
In a Brownian time series there is no correlation between the observations and a future observation. Series of this kind are hard to predict. A fractal dimension close to 1.5 ($H = 0.5$) is indicative of a Brownian time series. In this case, the market follows the efficient market hypothesis (EMH).

(2) **An anti-persistent time series**:
In an anti-persistent time series (also known as a mean-reverting series) an increase will most likely be followed by a decrease or vice-versa (i.e., values will tend to revert to a mean). This means that future values have a tendency to return to a long-term mean. A fractal dimension between 1.5 and 2 ($0 < H < 0.5$) is indicative of anti-persistent behavior and the closer the value is to 2, the stronger is the tendency for the time series to revert to its long-term means value. In this case market exhibits more volatile behavior than accommodated by EMH . When $H << 0.5$, the market exhibits ergodicity strongly disconfirming EMH.

(3) **A persistent time series**:
In a persistent time series an increase in values will most likely be followed by an increase in the short term and a decrease in values will most likely be followed by another decrease in the short term. A fractal dimension between 1 and 1.5 ($0.5 < H < 1$) indicates persistent behavior; the smaller the $D$ value the stronger the trend.However when $H \approx 1$, there is high risk of abrupt changes.

**Volatility** is one of the most important factors when pricing an asset. Volatility Index (VIX) is considered to be a premier barometer of investor's sentiment and market volatility. It is often described as the rate and magnitude of changes in prices. The importance of volatility forecasting was highlighted when in 2003 an American economist Robert Fry Engle III and British economist Clive Granger, shared a Noble prize for methods of analyzing economic time series with time - varying volatility. The goal of any volatility model is to be able to forecast volatility as accurate as possible. The 1996 and 1999 Basel Accord makes it compulsory for financial institutions to incorporate financial risk exposure in calculating the basic capital requirements. This makes volatility forecasting an obligatory task for all financial institutions.

Motivated by climate predictability index in Atmospheric Science [6, 7], here we suggest predictability index for stock market parameters [1]. As a first example, we consider the volatility of the market. We define volatility of the market as a quadruple vector

$$\mathbf{V} = (V_O, V_H, V_L, V_C),$$

where $V_O$ = volatility at the opening of the day, $V_H$ = highest volatility during the day, $V_L$ = lowest volatility during the day and $V_C$ = volatility at the closing of the day. Assuming each of these parameters $V_O, V_H, V_L, V_C$ following fBM, the predictability index vector of volatility is defined to be the quadruple

$$PI_V = (PI_O, PI_H, PI_L, PI_C),$$

where $PI_O = 2|D_O - 1.5|, PI_H = 2|D_H - 1.5|, PI_L = 2|D_L - 1.5|$ and $PI_C = 2|D_C - 1.5|$. Here $D_O, D_H, D_L, D_C$ are fractal dimensions of the time series $V_O, V_H, V_L, V_C$ respectively.

For India VIX we have analyzed total 1000 data point from 1st June 2009 to 29th May 2013.

TABLE 1. For India VIX

| Opening VIX $H_O$ | High VIX $H_H$ | Low VIX $H_L$ | Closing VIX $H_C$ |
|---|---|---|---|
| 0.8878 | 0.8914 | 0.8873 | 0.8909 |
| $D_O$ 1.1122 | $D_H$ 1.1086 | $D_L$ 1.1127 | $D_C$ 1.1091 |
| $PI_O$ 0.7756 | $PI_H$ 0.7828 | $PI_L$ 0.7746 | $PI_C$ 0.7818 |

Predictability Index vector

$$PI_V = (0.7756, 0.7828, 0.7746, 0.7818)$$

Thus, India VIX exhibits highly persistent behavior.

(1) The calculated values of fractal dimensions reveals that each of the time series $V_O, V_H, V_L, V_C$ follows fractional Brownian motions exhibiting persistence behavior in conformity with long range memory and resulting into non zero predictability attribute.

(2) A comparison of predictability indices for CBOE VIX (S & P 500 Index) and India VIX (NIFTY) reveals that a more matured market like US market exhibit more predictable behavior than a complex Indian Market.

We have applied the concept of predictability index to the data of FII (Foreign Institutional Investors) inflows, IIP (Index of Industrial Production) numbers, CPI (Consumer Price Inflation), Global Index (Consideration of Dow Jones Index). We infer that the CPI data exhibited predictable nature. IIP and FII data not predictable. Also, we have observed that BSE Index depend on FII in the shorter run.

We have analyzed different stock market indices for 200 days. Good amount of predictability found in Shanghai Composite. Little Predictability found in Dow Jones and Straight Time Index. BSE, Nifty, Nikkei approximates the Brownian motion and are therefore unpredictable following a random walk. we have also analyzed currencies of different Asian countries against US$ for last one year movement. The currencies of Saudi Arabia, China exhibit anti persistent behavior indicating either more stability or more volatility; whereas those of Malaysia, Philippines, Singapore more or less follow random walk. Lastly, we have also analyzed the data of Gold prices for 30 years price movement. We observed that 30 years daily closing price of gold for follow more or less a Brownian path. The data analysis suggest that the PI based on fractal dimension analysis in the frame work of a financial market vector is a promising tool to understand the dynamics of the market.

## My Research Papers with Professor Bhatt

(1) *Fractional Brownian motion and Predictability index in Financial Market*, (jointly with H. V. Dedania), Global Journal of Mathematical Sciences, 5(3)(2013) 197 - 203.

(2) *Fractal Dimensional Analysis in Financial Time Series*, (jointly with H. V. Dedania), International Journal of Financial Management, 5(2)(2015) 57-62.

## References

[1] S. J. Bhatt, H.V. Dedania and V. R. Shah, *Fractional Brownian motion and Predictability index in Financial Market*, Global Journal of Mathematical Sciences, 5(3)(2013) 197 - 203.

[2] S. J. Bhatt, H.V. Dedania and V. R. Shah, *Fractal Dimensional Analysis in Financial Time Series*, International Journal of Financial Management, 5(2)(2015) 57-62.

[3] B. B. Mandelbrot and J. W. van Ness, *Fractional Brownian Motions, Fractional Noises and Applications*, SIAM Review, 10(4)(1968) 422 - 437.

[4] R. F. Mulligen, *Fractal Analysis of Highly Volatile Markets : an Application to Technology Equities*,The Quarterly Review of Economics and Finance, 44(1)(2004)155 - 179.

[5] Bo Qian, Khaled Rasheed, *Hurst Exponent and Financial Market Predictability*, IASTED conference on "Financial Engineering and Applications",(2004) 203-209.

[6] G. Rangarajan, D. A. Sant, *A Climate Predictability Index and its Applications*, Geophysical Research Letters, 24(10)(1997) 1239 - 1242.

[7] G. Rangarajan, D. A. Sant, *Fractal Dimensional Analysis of Indian Climatic Dynamics*, Chaos, Solitons and Fractals 19(2004) 285 - 291.

[8] S. Rostek, *Option Pricing in Fractional Brownian Markets*, 622 Lecture Notes in Economics & Mathematical Systems, Springer-Verlag, Berlin Heidelberg, New York, 2009.

Applied Science and Humanities Department, G H Patel College of Engineering & Technology, Vallabh Vidyanagar, 388120 (India)

*E-mail address*: vipulshah@gcet.ac.in